CL=CSCI 160

CLASS 5

1.24 Formulate a weighted binary code for the decimal digits, using weights

a) *6, 3, 1, 1
b) 6, 4, 2, 1

Solution

	6 3 1 1	6 4 2 1
0	0 0 0 0	0 0 0 0
1	0 0 0 1(*	0 0 0 1
2	0 0 1 1	0 0 1 0
3	0 1 0 0	0 0 1 1
4	0 1 1 0(*	0 1 0 0
5	0 1 1 1	0 1 0 1
6	1 0 0 0	0 1 1 0(*
7	1 0 1 0(*	1 0 0 1(*
8	1 0 1 1	1 0 1 0
9	1 1 0 0	1 0 1 1

(* Here we have a choice

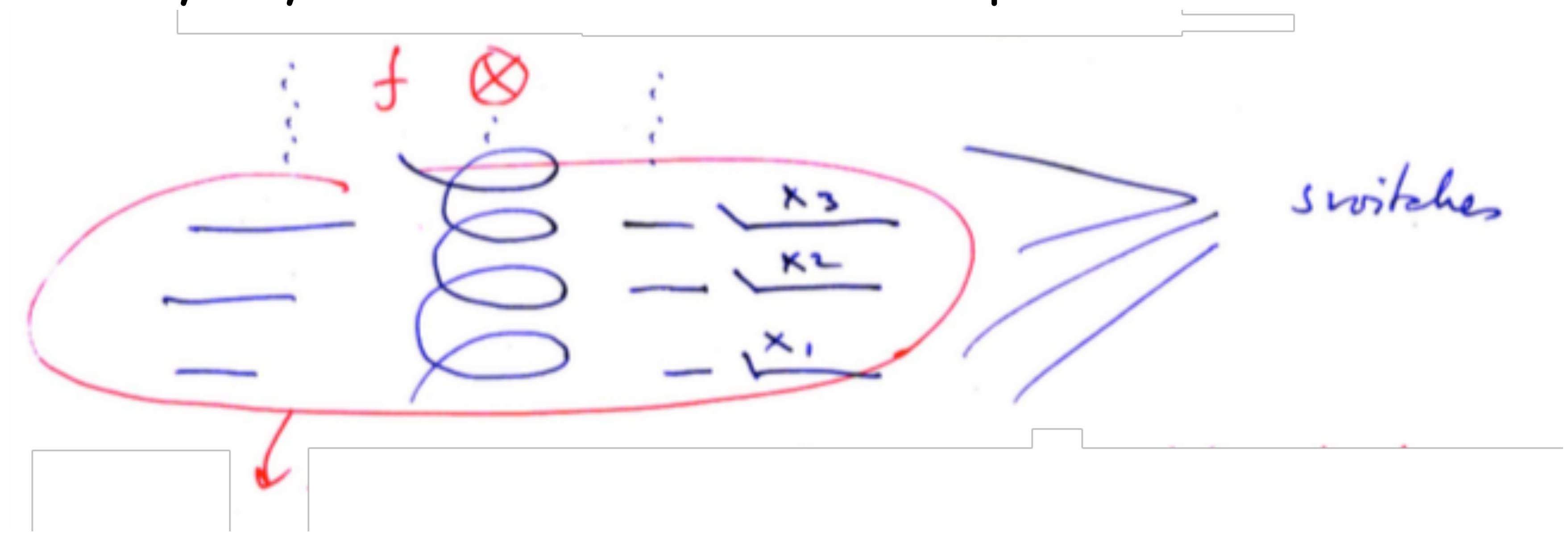
Table
Four Different Binary Codes for the Decimal Digits

Decimal Digit	BCD 8421	2421	Excess-3	8, 4, −2, −1
0	0000	0000	0011	0000
1	0001	0001	0100	0111
2	0010	0010	0101	0110
3	0011	0011	0110	0101
4	0100	0100	0111	0100
5	0101	1011	1000	1011
6	0110	1100	1001	1010
7	0111	1101	1010	1001
8	1000	1110	1011	1000
9	1001	1111	1100	1111
	1010	0101	0000	0001
Unused	1011	0110	0001	0010
bit	1100	0111	0010	0011
combi-	1101	1000	1101	1100
nations	1110	1001	1110	1101
	1111	1010	1111	1110

Note None of these codes is **self-complementary**, i.e. ((r-1)s complements correspond to each other, that is decimal digits complementary in 9's compl. have binary representations that are 1's complements of each other.

Problem of the Independent Switches: example of a Boolean function

There is a staircase leading to **n** floors, all being lit by one lamp, which is independently operated by any of **n** switches, which are placed one at each of the **n** floors.



To simplify, consider **n = 3**; that is there are **3** floors and **3 switches**.

Design a function **f** of the 3 switches. **f** represents this lamp. Find its truth table.

x_1	x_2	x_3	f_1	f_2
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

Say: on = 1 and off = 0

Note: the value of **f** for any initial state of the switches is arbitrary, as we can start with any configuration; however once we choose one initial state, the values of **f** for every other configuration of the switches are determined by this initial state.

f_1 and f_2 are **complementary**

Hamming Distance (d_H) between two strings equals the number of locations in which the two strings differ.

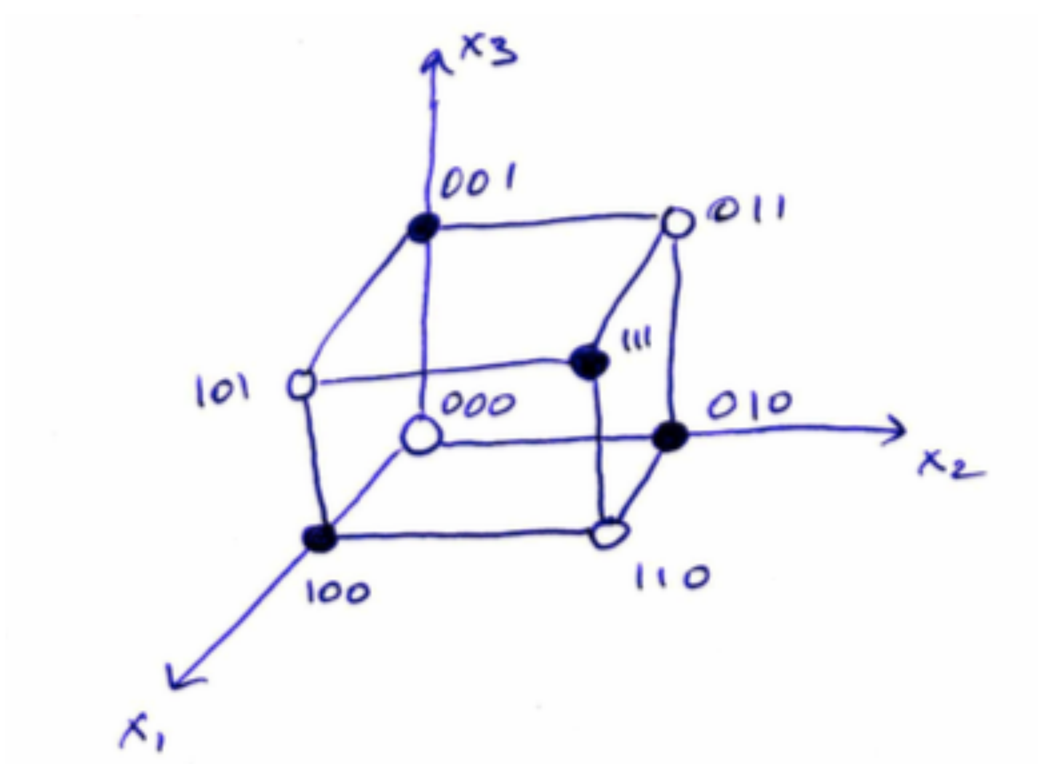
ex: $d_H(1\,\underline{0}\,0,\ 1\,\underline{1}\,0) = \mathbf{1}$; $d_H(\underline{0}\,\underline{0}\,\underline{1},\ \underline{1}\,\underline{1}\,\underline{0}) = \mathbf{3}$

x_1	x_2	x_3	f_1	f_2
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

Hamming Distance (d_H) between two strings equals the number of locations in which the two strings differ.

Note: When the H distance between two $x_1\, x_2\, x_3$ values is **odd** then the corresponding values for **f** are **different**. Conversely, when the H distance is **even** then the corresponding values for **f** are **the same**.

Let's represent the problem of the 3 independent switches using a 3-D cube.



The vertices of the cube have as coordinates all possible inputs for the 3 switches, while the color of each vertex shows the state of the lamp (**f**).

● = on <--> **f** = 1; ○ = off <--> **f** = 0

HW: Solve the problem of the independent switches for **n = 4**, which means:

1) Truth table
2) Draw the 4-D cube

Boolean Algebra - Definition

A Boolean algebra is a 6-tuple: $(B, +, \cdot, ' ; 0, 1)$

B: Set; +: or; $\cdot$: and; ': complement; 0: identity elem. for +; 1: identity elem. for $\cdot$

that follows the next 6 axioms:

1) **Closure** for +, $\cdot$
   if $x, y \in B$, then $x \cdot y,\ x + y \in B,\ \forall x, y \in B$
2) **Identity elements**:
   $x + 0 = 0 + x = x$
   $x \cdot 1 = 1 \cdot x = x,\ \forall x \in B$
3) **+, $\cdot$ Commutative**:
   $x + y = y + x$
   $x \cdot y = y \cdot x, \forall x, y \in B$
4) **+, $\cdot$ Distributive**: one relative to the other
   $(x \cdot y) + z = (x + z)(y + z)$
   $(x + y) \cdot z = x \cdot z + y \cdot z \quad \forall x,y,z \in B$
5) $\forall x \in B, \exists x'$ or $\overline{x} \in B$ such that:
   $x + x' = 1$
   $x \cdot x' = 0$
6) **There are at least 2 elements**:
   $0 \neq 1$.

HW: Write all axioms and properties for the B. algebra of sets:

S = set, $S \neq \varnothing$

$(P(S), \cup, \cap, \complement; \varnothing, S)$.

P(S): Set of all subsets of S

[Remember: if S has n elements then P(S) has 2^n]

Axioms of the Boolean Algebra (B, + , · ,' ; 0,1)

1) +, · are **closed operations in B:**
   $x + y \in B$
   $x \cdot y \in B, \quad \forall x, y \in B$

2) **identity elements:**
   $x + 0 = 0 + x = x$
   $x \cdot 1 = 1 \cdot x = x, \quad \forall x \in B$

3) +, · are **commutative:**
   $x + y = y + x$
   $x \cdot y = y \cdot x, \ \forall x, y \in B$

4) +, · are **distributive** one relative to the other one:
   $(x + y) \cdot z = x \cdot z + y \cdot z$
   $(x \cdot y) + z = (x + z)(y + z), \quad \forall x, y, z \in B$

5) $\forall x \in B, \exists x' \in B$, the **complement** of x, such that:
   $x + x' = 1$ and
   $x\ x' = 0$

6) There are at least two elements in B:
   $0 \neq 1$

Properties of Boolean Algebras (B)

7) $+, \cdot$ are **idempotent:**
$x + x = x$
$x \cdot x = x, \quad \forall x \in B$

8) $x + 1 = 1,$
$x \cdot 0 = 0, \quad \forall x \in B$

9) $x'' = x, \quad \forall x \in B$

10) **DeMorgan's Law**:
$(x + y)' = x'y',$
$(x \cdot y)' = x' + y', \quad \forall x, y \in B$

11) $+, \cdot$ are **associative:**
$x + (y + z) = (x + y) + z,$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad \forall x, y, z \in B$

12) **Absorption** for $+, \cdot$:
$x + xy = x$
$x (x + y) = x, \quad \forall x, y \in B$

Axioms + Properties for Boolean Algebra (B, + , · , ' ; 0,1)

1) +, · are **closed in B**

2) **identity elements:**

   $x + 0 = x = 1 \cdot x, \quad \forall x$

3) +, · are **commutative**

4) +, · are **distributive** one relative to the other one:

   $(x + y) \cdot z = x \cdot z + y \cdot z$

   $(x \cdot y) + z = (x + z)(y + z), \quad \forall x, y, z$

5) $\forall x \in B, \exists x' \in B$, the **complement** of x, such that:

   $x + x' = 1$ and $\quad x \cdot x' = 0$

6) **0 ≠ 1**

7) +, · are **idempotent:**

   $x + x = x$ and $x \cdot x = x, \quad \forall x$

8) $x + 1 = 1$ and $x \cdot 0 = 0, \quad \forall x$ **Identity, Annulment**

9) $x'' = x, \quad \forall x$ **Double Negation**

10) **DeMorgan's Law**:

   $(x + y)' = x'y'$, and $(x \cdot y)' = x' + y', \quad \forall x, y$

11) +, · are **associative**

12) **Absorption** for +, · :

   $x + xy = x$

   $x (x + y) = x, \quad \forall x, y$

HW 3-D

The solutions to the quadratic equation

$x^2 - 11x + 22 = 0$

are $x = 3$ and $x = 6$.

Determine the base of the numbers in the equation.

Solution

use sol. $x = 3$: $3^2 - 11_r \cdot 3 + 22_r = 0 \overset{\text{base 10}}{\iff}$

$\iff 9 - (r+1)3 + 2r + 2 = 0 \iff r = 8$

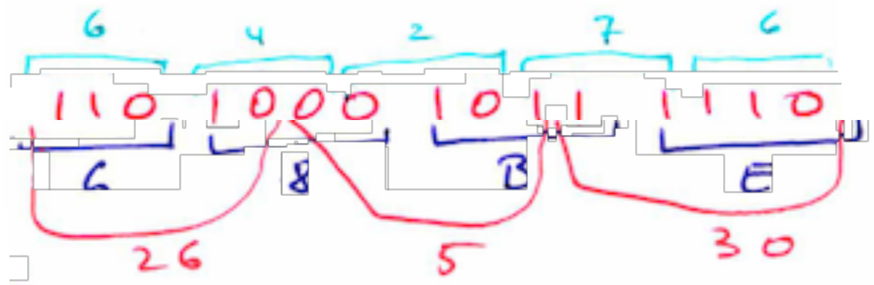
[one could use other solution $r = 6$ and obtain the same].

HW 3-E Convert the hexadecimal number 68BE to binary, and then convert it from binary to octal and then to base 32.

Solution

digit	r = 32
15	F
16	G
17	H
18	I
19	J
20	K
21	L
22	M
23	N
24	O
25	P
26	Q
27	R
28	S
29	T
30	U
31	V

$68BE_{hex} = 110\ 1000\ 1011\ 1110$



$= 64276_8$

$= Q5U_{32}$