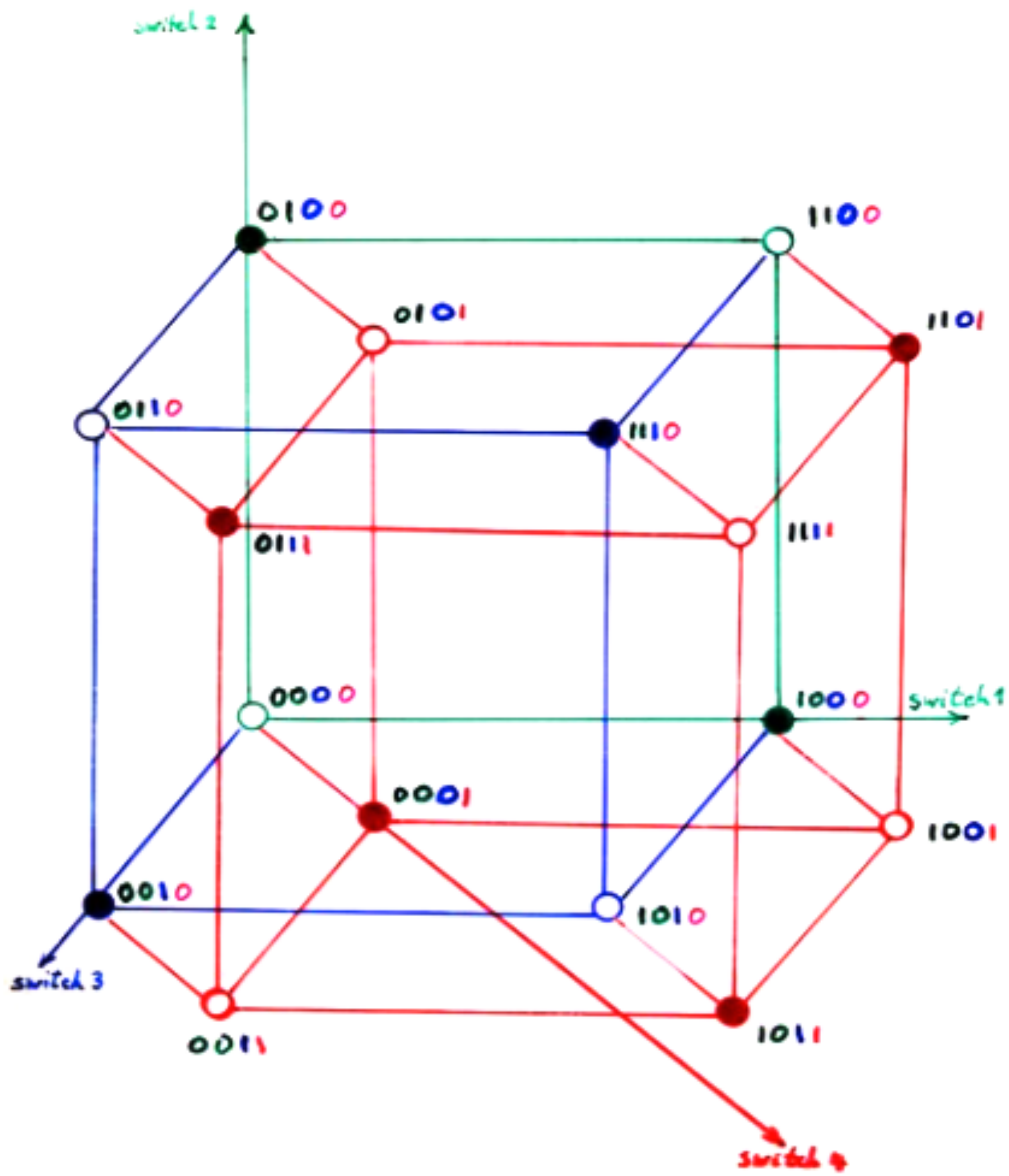
CL=CSCI 160

CLASS 6

HW: 4 independent switches-sol. (4-cube)

Solution 4-D cube



2-D cube

1-D cube

3-D cube

4-D cube

Solution (continued)

Truth Table 4 switches imply 4 input variables:

X_1	X_2	X_3	X_4	f_1	f_2
0	0	0	0	0	1
0	0	0	1	1	0
0	0	1	0	1	0
0	0	1	1	0	1
0	1	0	0	1	0
0	1	0	1	0	1
0	1	1	0	0	1
0	1	1	1	1	0
1	0	0	0	1	0
1	0	0	1	0	1
1	0	1	0	0	1
1	0	1	1	1	0
1	1	0	0	0	1
1	1	0	1	1	0
1	1	1	0	1	0
1	1	1	1	0	1

HW:

Write all axioms for the Boolean algebra (abrev. B.A.) of the power set of a set $S \neq \varnothing$, namely:

$$(P(S), \cup, \cap, \complement; \varnothing, S), \text{ where } P(S) = \{ X: X \subseteq S\}.$$

Solution

$(B, +, \cdot, '; 0, 1)$:

1) $+, \cdot$ are **closed operations in B**:
   $x + y \in B$
   $x \cdot y \in B, \forall x, y \in B$

2) **Identity elements**:
   $x + 0 = 0 + x = x$
   $x \cdot 1 = 1 \cdot x = x, \forall x \in B$

3) $+, \cdot$ are **commutative**:
   $x + y = y + x$
   $x \cdot y = y \cdot x, \forall x, y \in B$

4) $+, \cdot$ are **distributive** relative to each other:
   $(x + y) \cdot z = x \cdot z + y \cdot z$
   $(x \cdot y) + z = (x + z)(y + z), \forall x, y, z \in B$

5) $\forall x \in B, \exists x' \in B$, the **complement** of x, such that:
   $x + x' = 1$ and
   $x x' = 0$

6) There are at least two elements in B, as:
   $0 \neq 1$

$(P(S), \cup, \cap, \complement; \varnothing, S)$:

1) $\cup, \cap$ are **closed** in $P(S)$:
   $X \cup Y \subseteq S$ [$\Leftrightarrow \in P(S)$], $X \cap Y \subseteq S, \forall X, Y \subseteq S$
   [Note $X \in P(S) \Leftrightarrow X \subseteq S$]

2) **Identity elements**: $X \cup \varnothing = \varnothing \cup X = X$
   $X \cap S = S \cap X = X, \forall X \subseteq S$.

3) $\cup, \cap$, are **commutative**:
   $X \cup Y = Y \cup X$,
   $X \cap Y = Y \cap X, \forall X, Y \subseteq S$

4) $\cup, \cap$, are **distributive** one relative to the other one:
   $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$
   $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z), \forall X, Y \subseteq S$

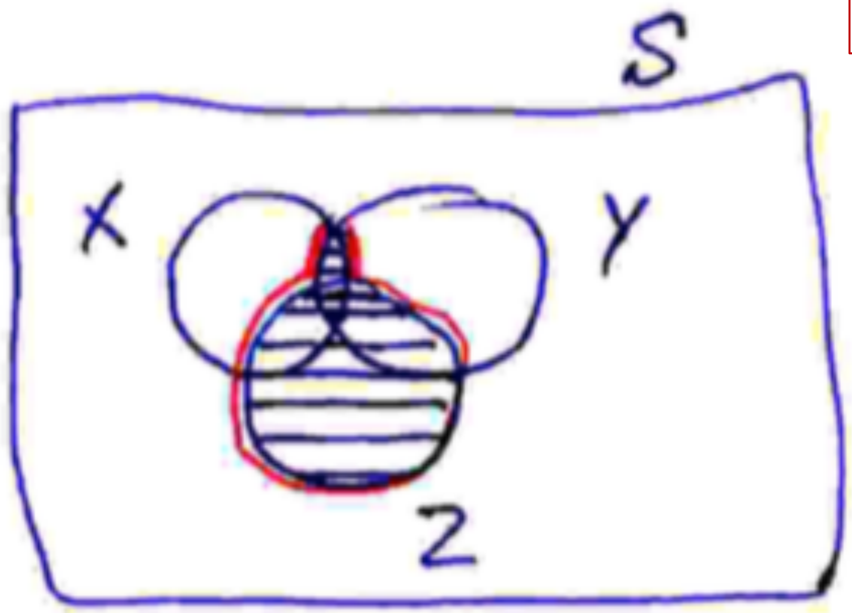
5) $\forall X \subseteq S, \exists \overline{X} \subseteq S$ (same as $\complement X$), called the **complement** of X, such that:
   $X \cup \overline{X} = S$, and $X \cap \overline{X} = \varnothing$.

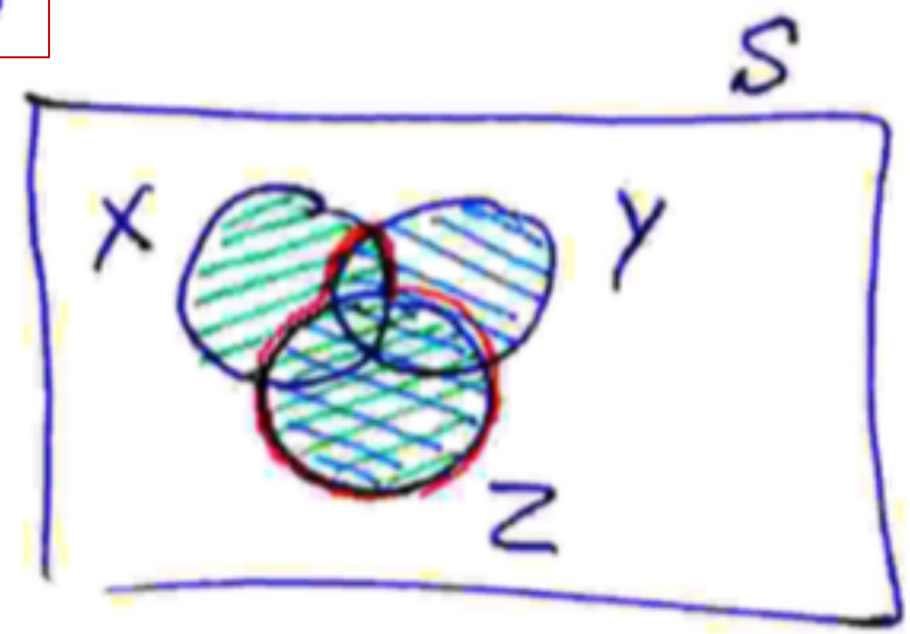
6) There are at least 2 elements in P(S), as we have:
   $\varnothing \neq S$
   for the identity elements.

Axioms 4) and 5)- Venn diagrams:

4) $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$.

LS = RS

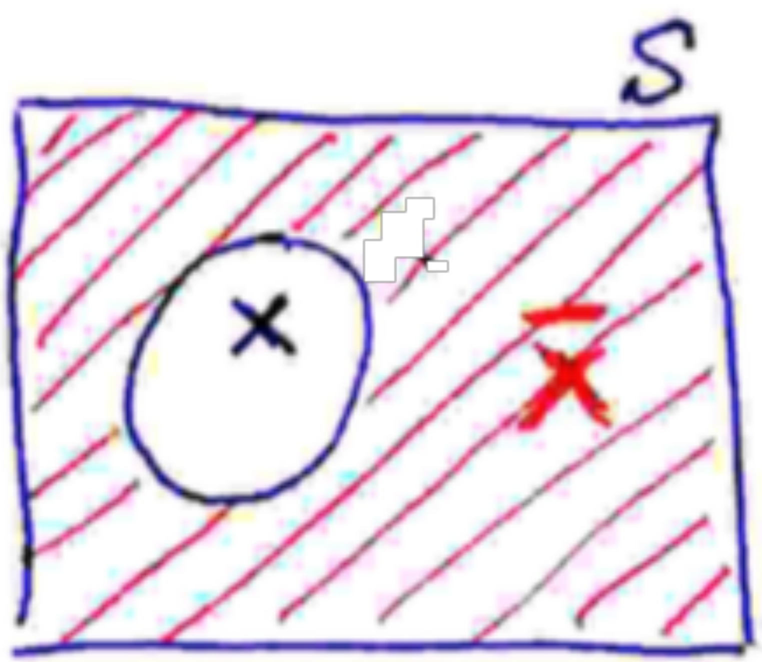




$(X \cap Y) \cup Z$ = LS

RS: $(X \cup Z) \cap (Y \cup Z)$ =

5)



$X \cup \overline{X} = S$, and $X \cap \overline{X} = \varnothing$.

HW 6.1 assigned:

Write all 7) - 12) properties for the algebra of the power set of a

set $S \neq \varnothing$, namely: $(P(S), \cup, \cap, \complement; \varnothing, S)$, where $P(S) = \{ X : X \subseteq S \}$,

if you haven't done this for today.

Theorem: Every Boolean algebra is equivalent to a B. A. of a power set for some set that is not empty.

⇨ Size of any Boolean algebra = 2^k, for some $k > 0$, where k is the size of the set, for which we take the power set that is the B. A.

Principle of duality:

If we have a true statement and swap:

$+ \longleftrightarrow \cdot$ and $0 \longleftrightarrow 1$

we obtain another true statement.

Next, for any Boolean algebra (B, +, $\cdot$, ', 0,1), where we know that the axioms 1) - 6) hold let's prove the properties 7 - 12.

Properties – to prove using Axioms 1) – 6):

2) **Identity elements:**
$x + 0 = 0 + x = x$
$x \cdot 1 = 1 \cdot x = x, \forall x \in B$

4) +, · are **distributive** relative to each other:
$(x + y) \cdot z = x \cdot z + y \cdot z$
$(x \cdot y) + z = (x + z)(y + z), \forall x, y, z \in B$

5) $\forall x \in B, \exists x' \in B$, the **complement** of x, such that:
$x + x' = 1$ and
$x x' = 0$

7) +, · are **idempotent**:

$x + x = x$

$x \cdot x = x, \forall x \in B$

By duality we only need to prove just one of these two equalities above.

Proof

$$\boxed{x} \overset{(2)}{=} x + 0 \overset{(5)}{=} x + x \cdot x' \overset{(4)}{=} (x + x) \cdot (x + x') \overset{(5)}{=} (x + x) \cdot 1 \overset{(2)}{=} \boxed{x + x}$$

Let's try to prove the dual equality via a dual proof, following the dual axioms.

It is not needed: we just do it as an exercise, to see duality in action.

$$\boxed{x} \overset{(2)}{=} x \cdot 1 \overset{(5)}{=} x \cdot (x + x') \overset{(4)}{=} x \cdot x + x \cdot x' \overset{(5)}{=} x \cdot x + 0 \overset{(2)}{=} \boxed{x \cdot x}$$

HW 6.2-assigned :

Prove all properties 8) – 12) using axioms 1) – 6) and the properties you just proved. **Attention**: For property 11), associativity, use truth tables on the variables and assume only the values of the 2-element B.A., namely B = {0, 1}, also called the switching algebra.

Exercises (Manipulation in B. A.)

$$xy + xy' \overset{(4)}{=} x(y + y') \overset{(5)}{=} x \cdot 1 \overset{(2)}{=} x$$

$$(x + y)(x + y') \overset{(4)}{=} x + yy' \overset{(5)}{=} x + 0 \overset{(2)}{=} x$$

$$zx + zx'y \overset{(4)}{=} z(x + x'y) \overset{(4)}{=} z(x + x')(x + y) \overset{(5)}{=} z \cdot 1 \cdot (x + y) \overset{(2)}{=} z(x + y)$$

$$(x + y)'(x' + y')' \underset{(9)}{\overset{(10)}{=}} x'y' \cdot x \cdot y \underset{(5)}{\overset{(3)}{=}} 0$$

10) **DeMorgan's Law**:
$(x + y)' = x'y'$,
$(x \cdot y)' = x' + y'$, $\forall x, y \in B$

9) $x'' = x$, $\forall x \in B$

Prove: $xyz + x'y'z + x'yz + xyz' + x'y'z' = xy + x'y' + yz$

Proof

$$LS = xyz + x'y'z + x'yz + xyz' + x'y'z' \overset{(7)}{=} xyz + xyz + x'y'z + x'yz + xyz' + x'y'z' \overset{(4)}{=}$$

$$= \underset{(5)}{xy(z + z')} + \underset{(5)}{(x + x')yz} + \underset{(5)}{x'y'(z + z')} = \underset{(2)}{xy \cdot 1} + \underset{(2)}{1 \cdot yz} + \underset{(2)}{x'y' \cdot 1} =$$

$$= xy + yz + x'y' = RS$$

Done!