Recursion

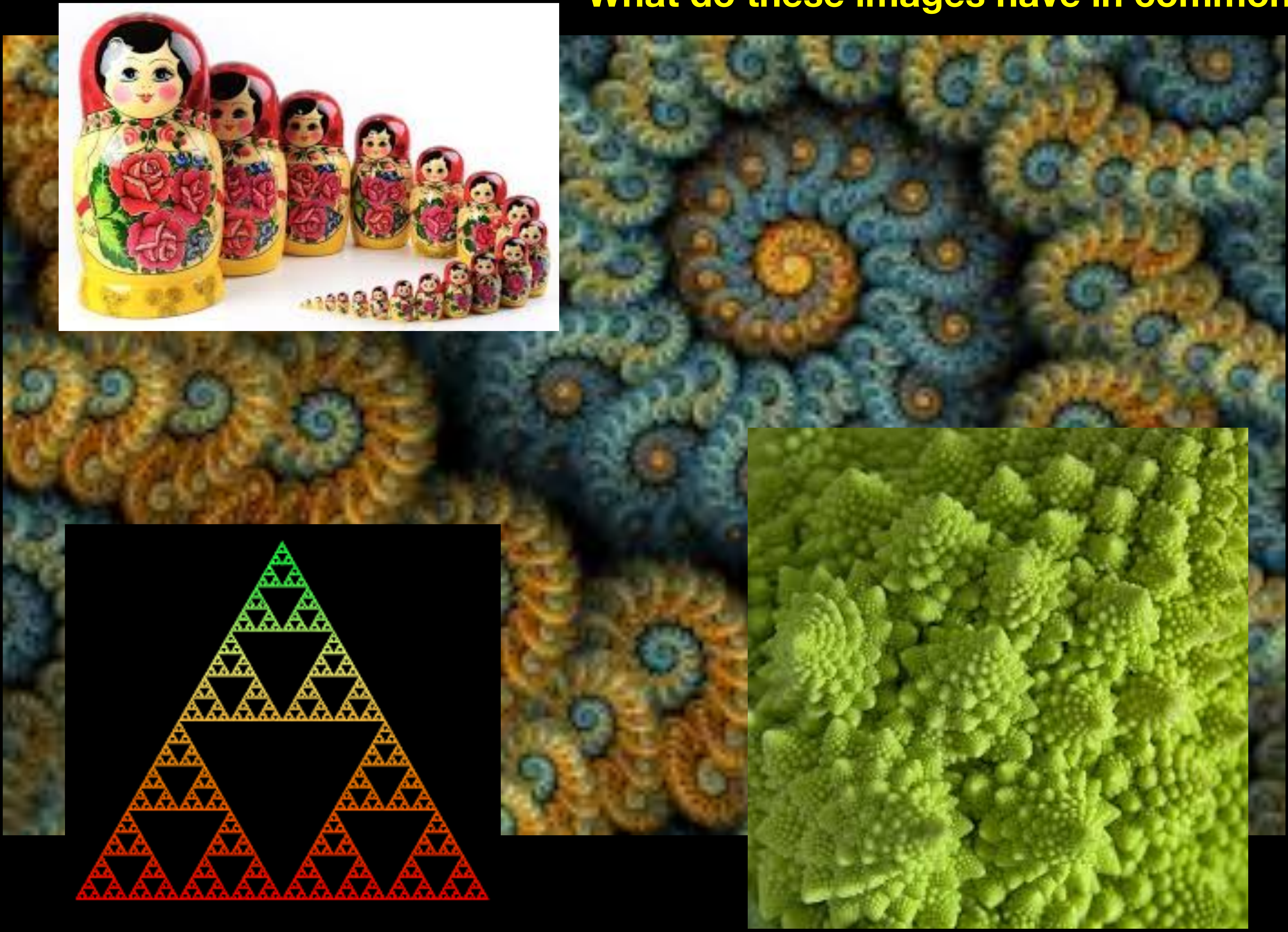
Tiziana Ligorio

Today’s Plan

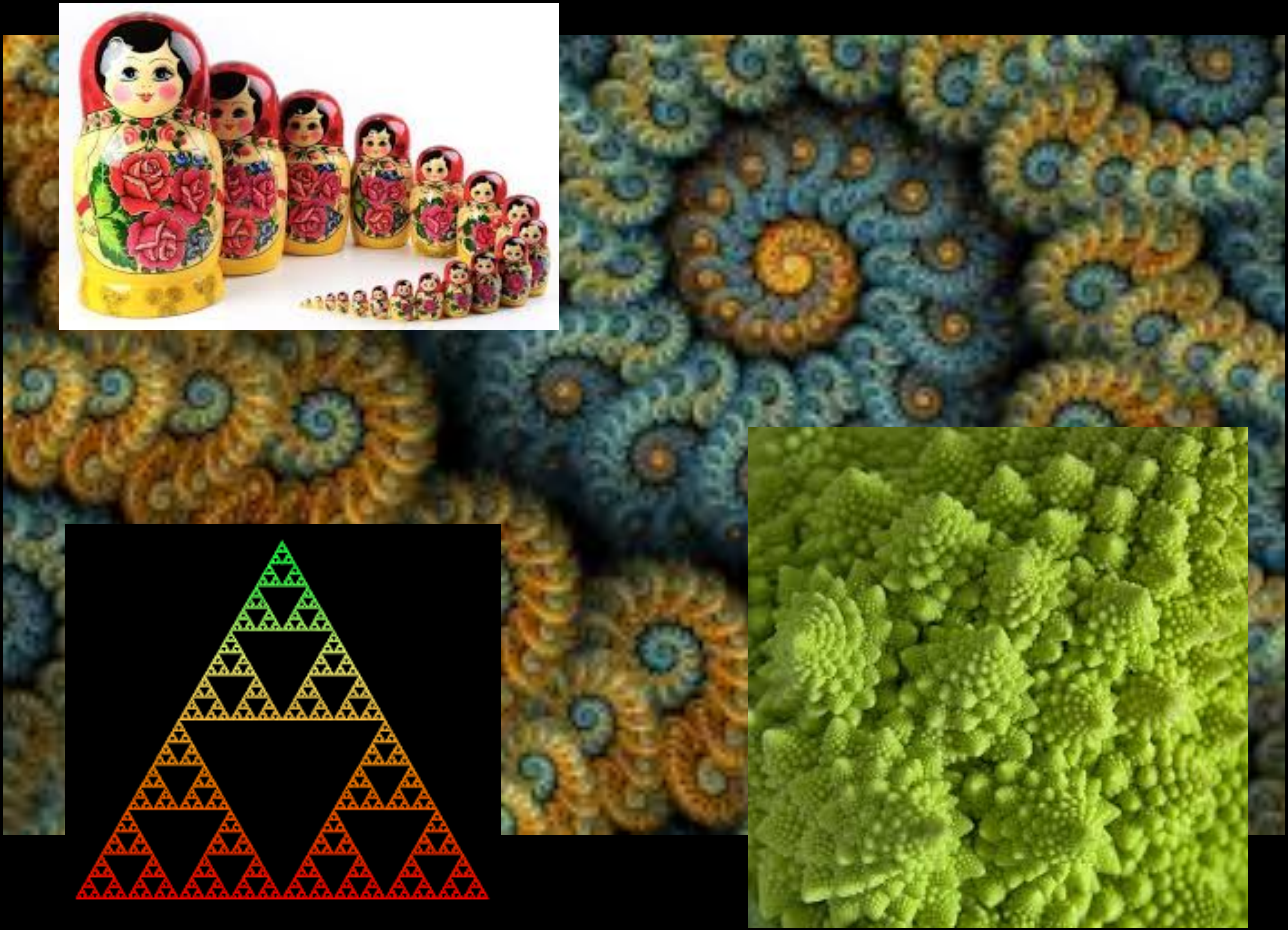
Announcements

Recursion

What do these images have in common



They contain a SMALLER copy of THEMSELVES



Print String Backwards

“Hello”

Print String Backwards

"Hello"

Procedure:

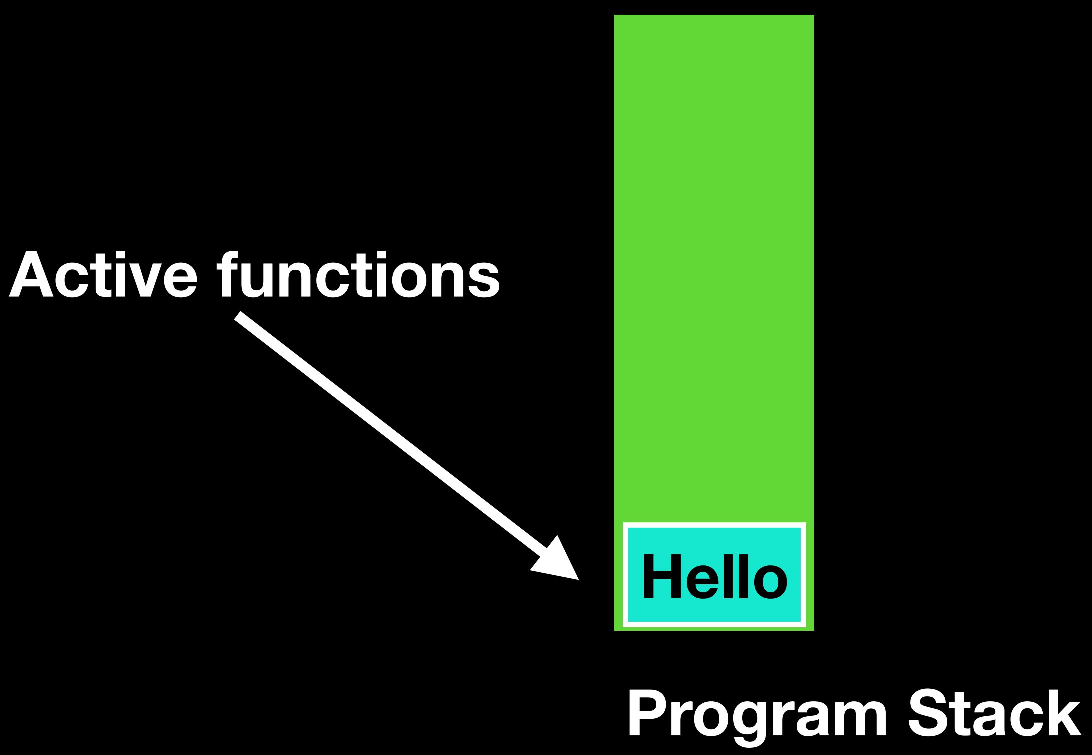
```
If there are characters to print
    Print the last character and reverse the rest
```

Recursive Call
Notice it's the last thing it does

Print String Backwards

Hello

o

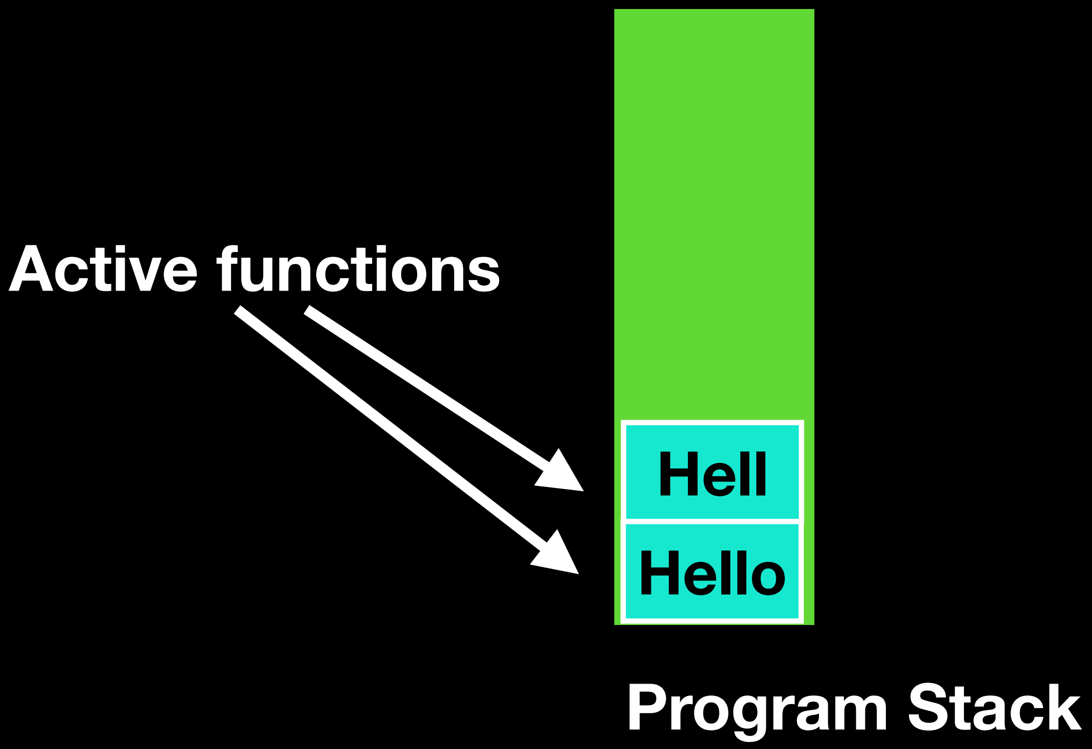


Print String Backwards

Hello

o

→ Hell



Active functions

Hell

Hello

Program Stack

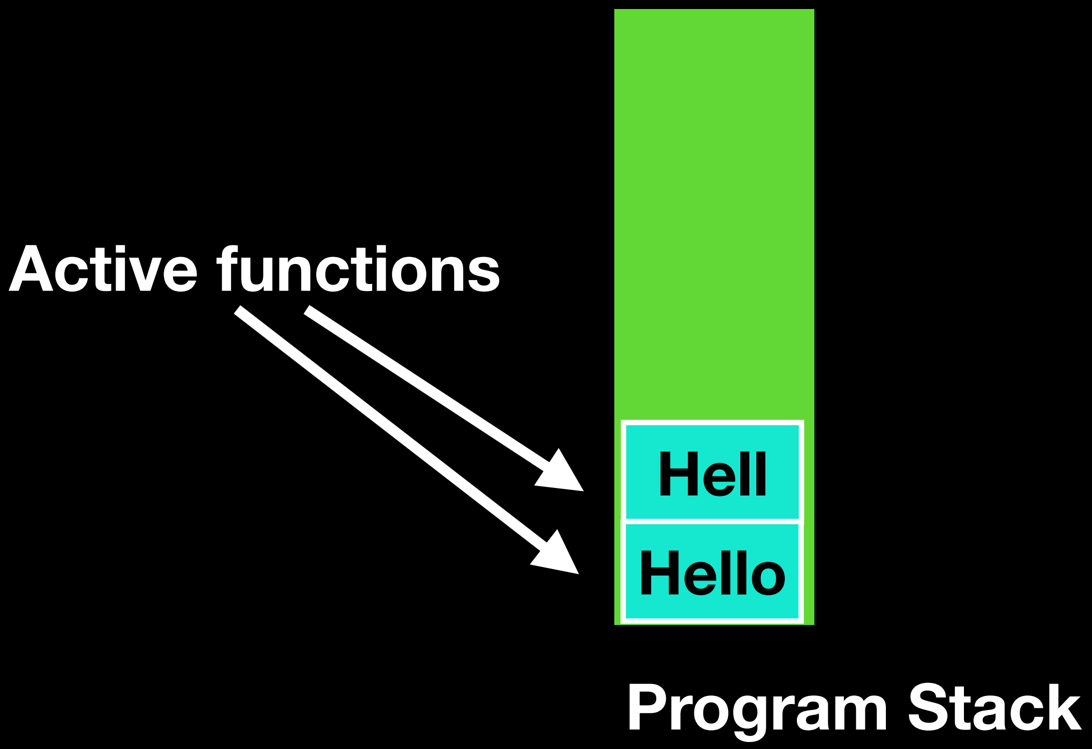
Print String Backwards

Hello

o

→ Hell

o l



Active functions

Hell

Hello

Program Stack

Print String Backwards

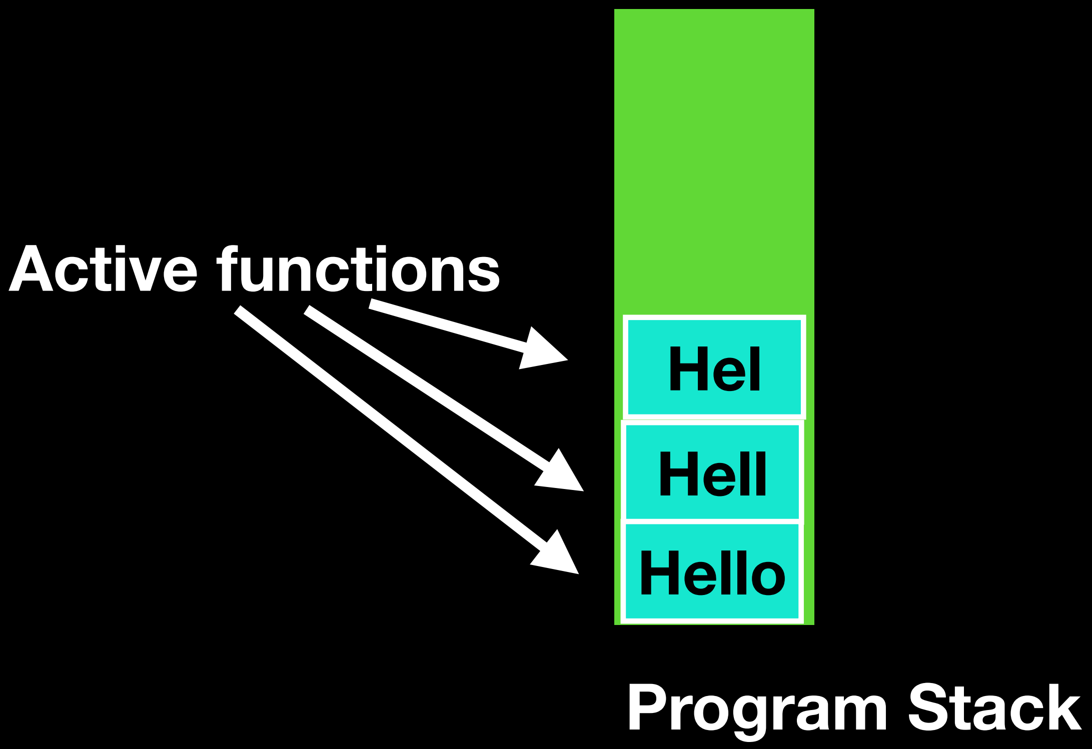
Hello

o

→ Hell

o l

→ Hel



Active functions

Program Stack

Print String Backwards

Hello

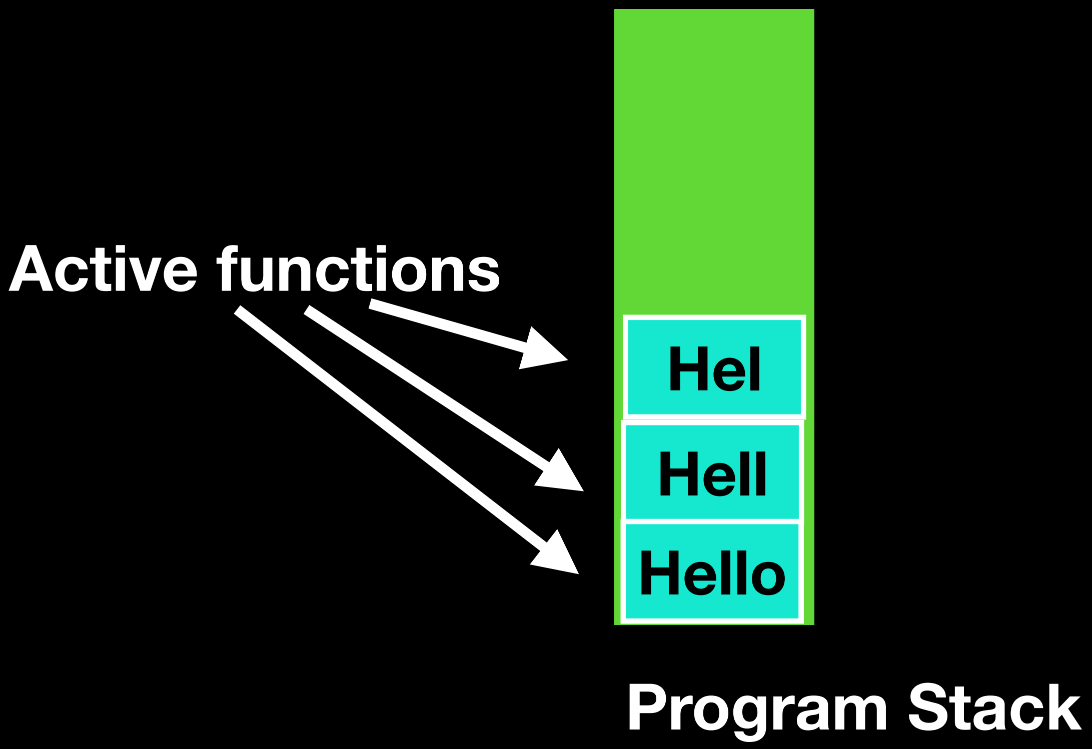
o

→ Hell

o l

→ Hel

o l l



Active functions

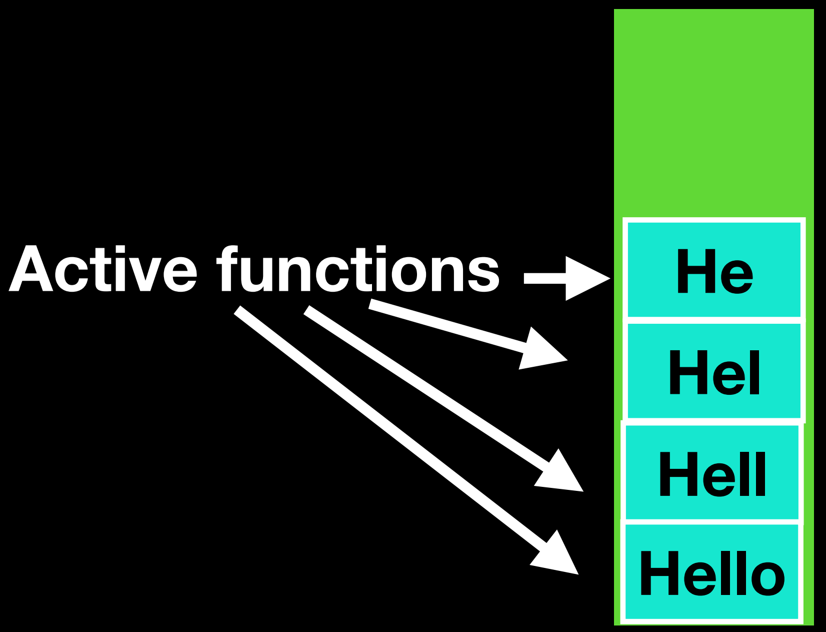
Program Stack

Print String Backwards

```
Hello
o
→ Hell
  o l

→     Hel
      o l l

→         He
```



Active functions

He

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

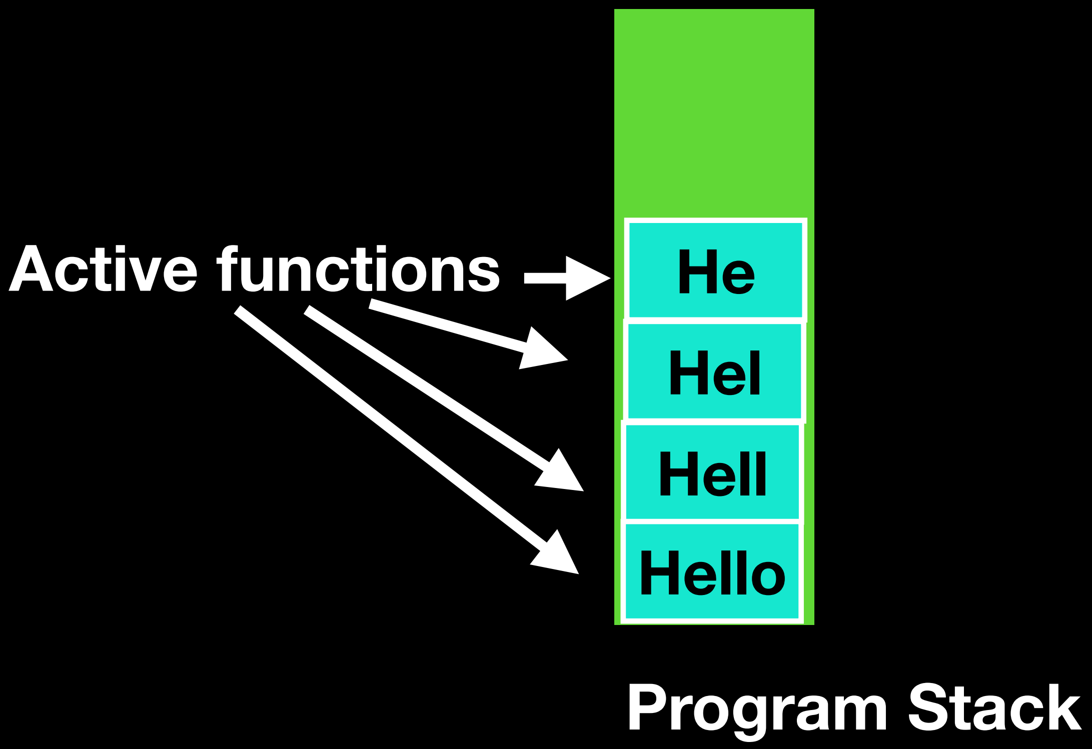
o l

→ Hel

o l l

→ He

o l l e



Print String Backwards

Hello

o

→ Hell

o l

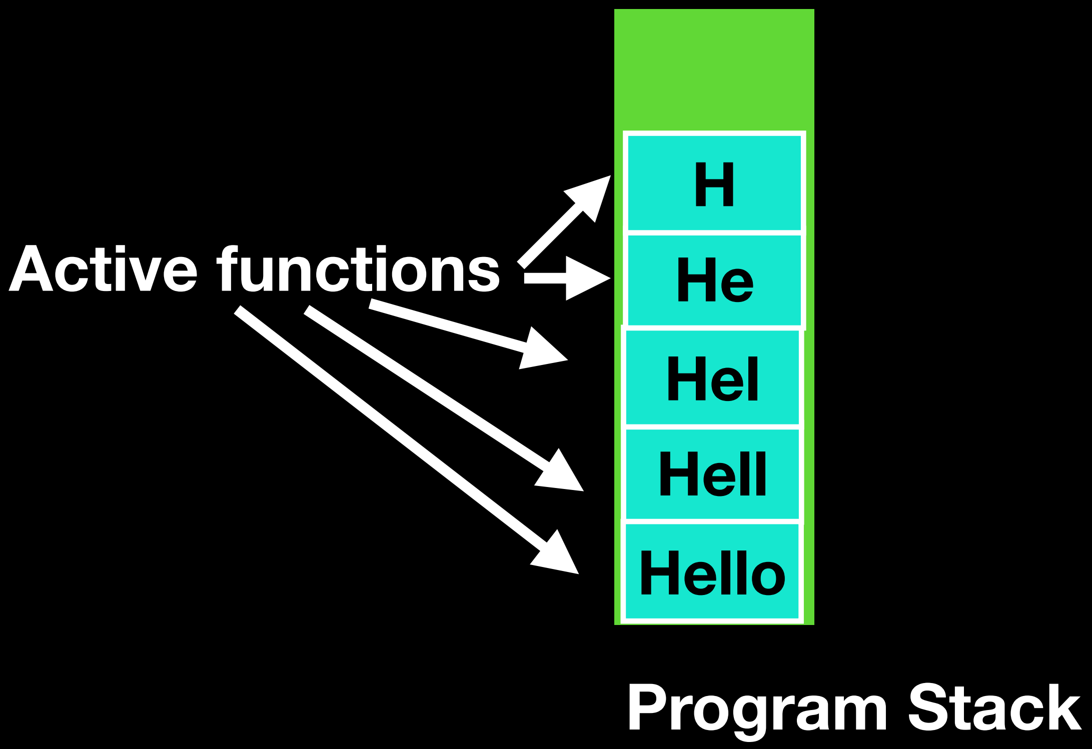
→ Hel

o l l

→ He

o l l e

→ H



Print String Backwards

Hello

o

→ Hell

o l

→ Hel

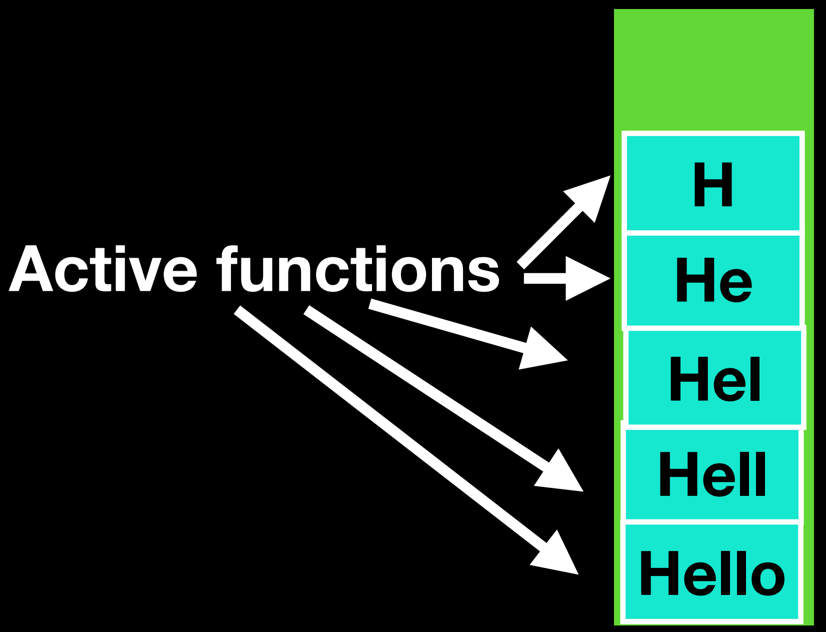
o l l

→ He

o l l e

→ H

o l l e H



Active functions

H

He

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

o l l

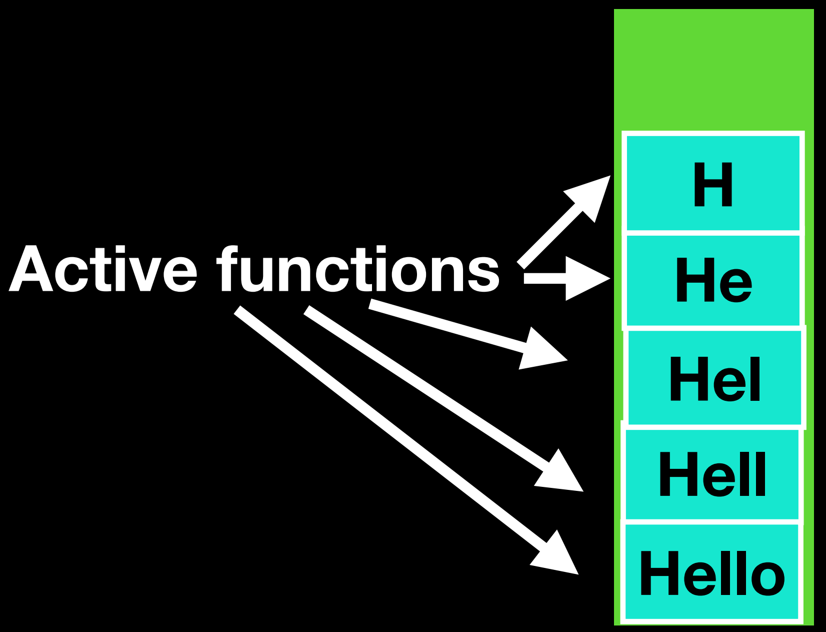
→ He

o l l e

→ H

o l l e H

BASE CASE



Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

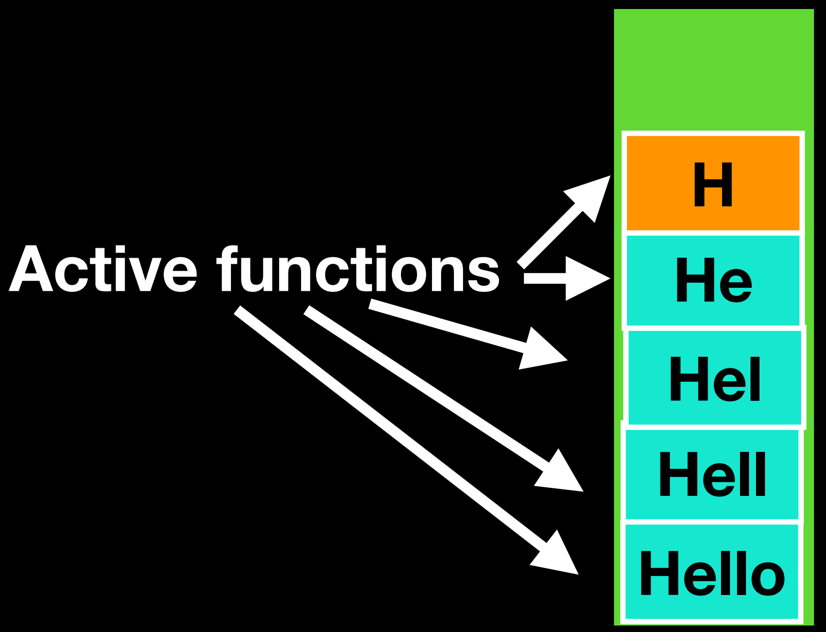
o l l

→ He

o l l e

→ H

o l l e H



Active functions

H

He

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

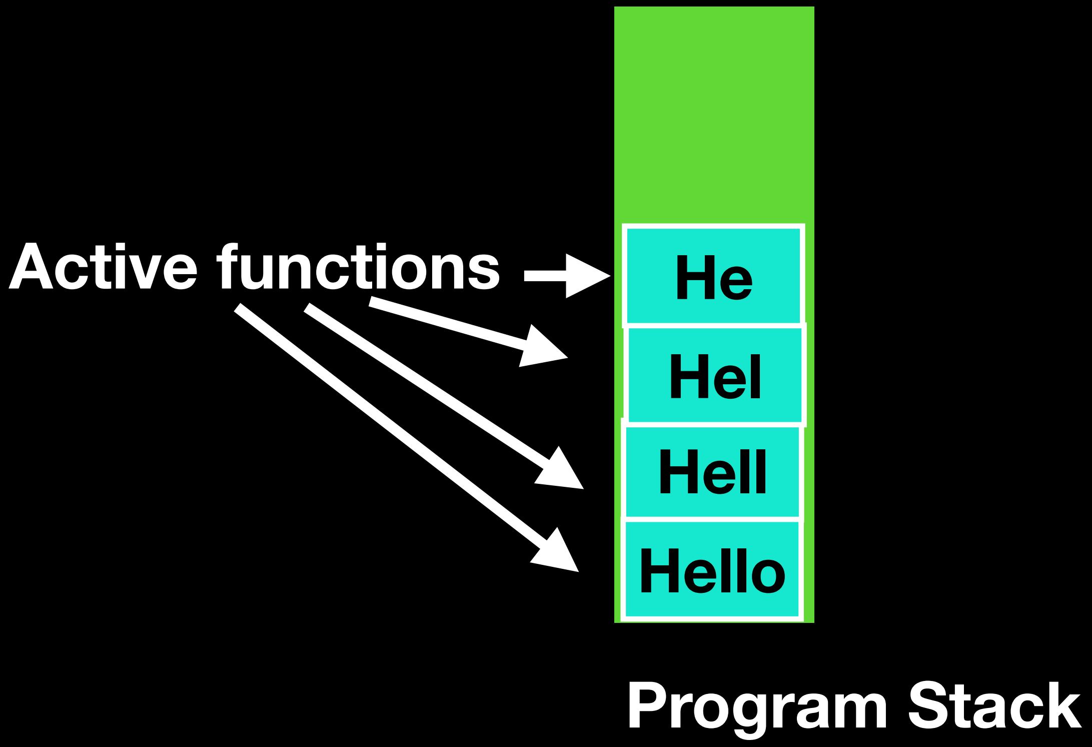
o l l

→ He

o l l e

→ H

o l l e H



Active functions

He

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

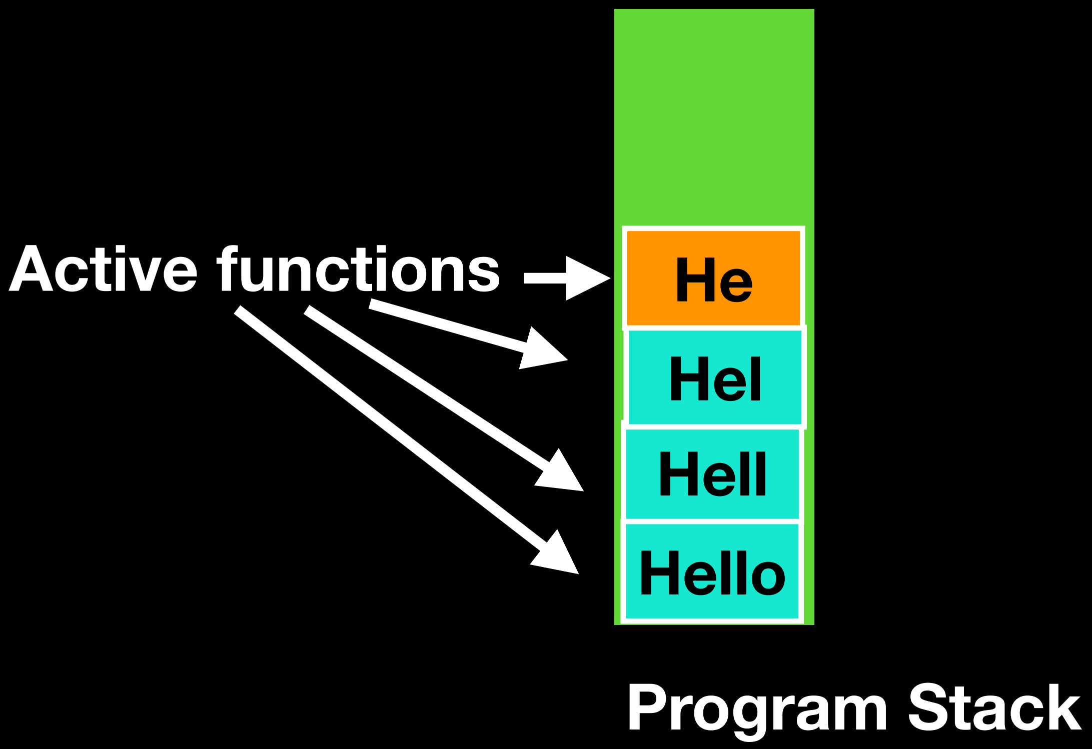
o l l

→ He

o l l e

→ H

o l l e H



Active functions

He

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

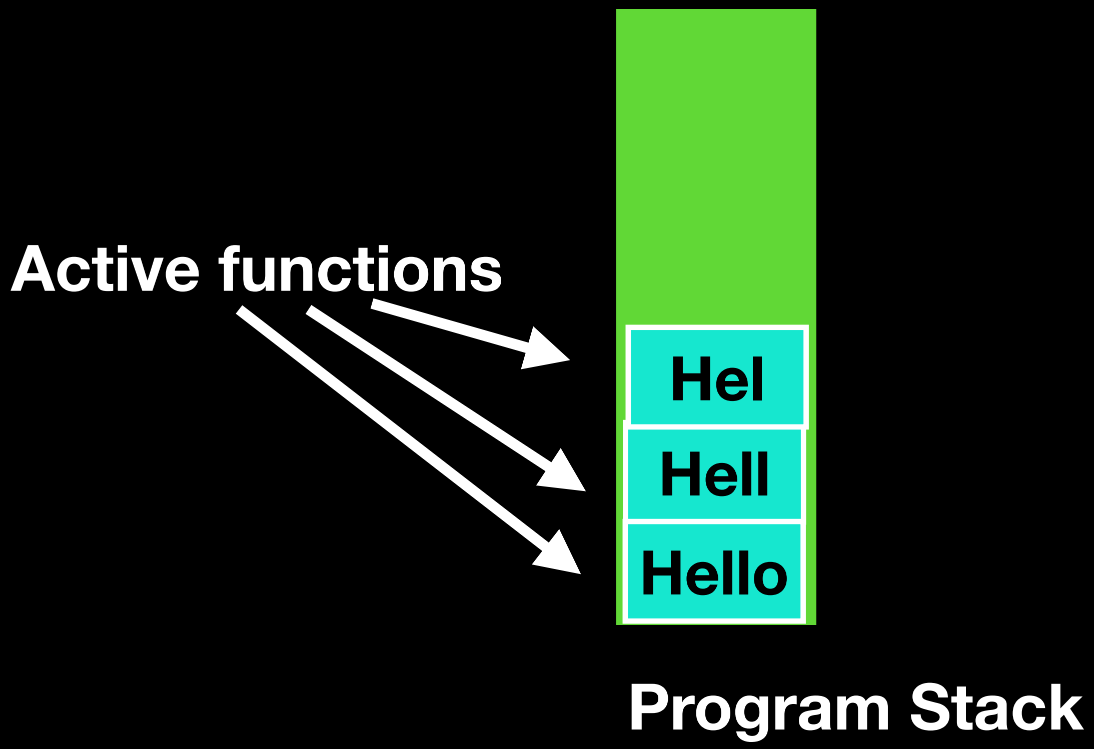
o l l

→ He

o l l e

→ H

o l l e H



Active functions

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

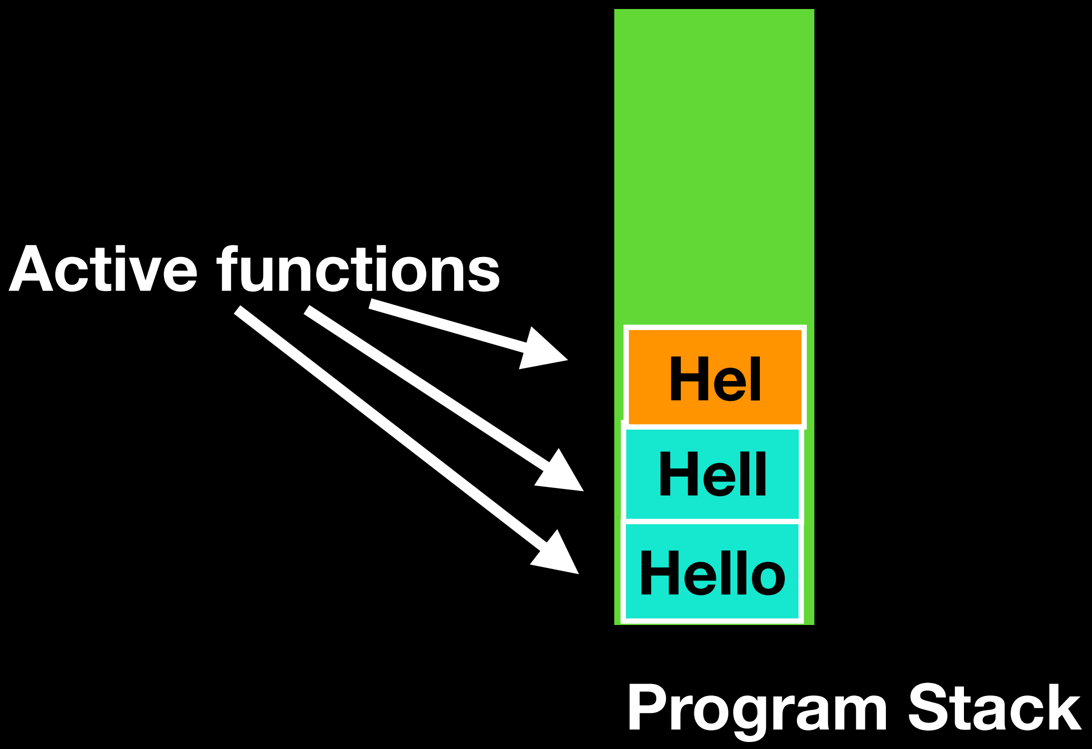
o l l

→ He

o l l e

→ H

o l l e H



Active functions

Hel

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

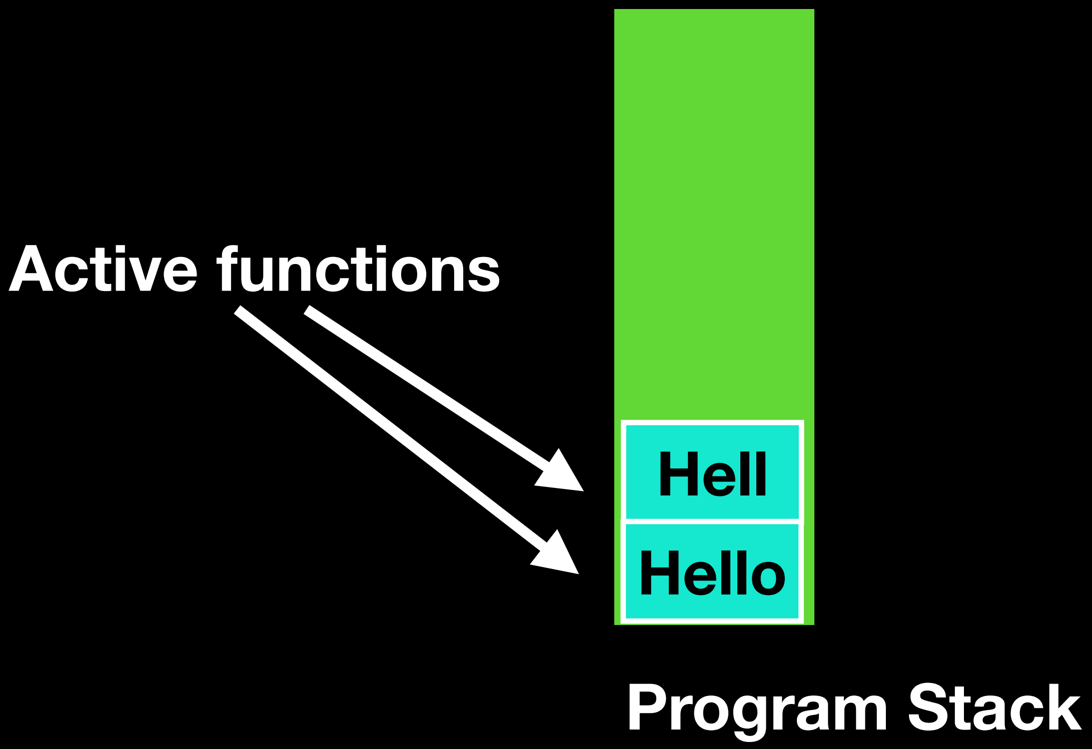
o l l

→ He

o l l e

→ H

o l l e H



Active functions

Hell

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

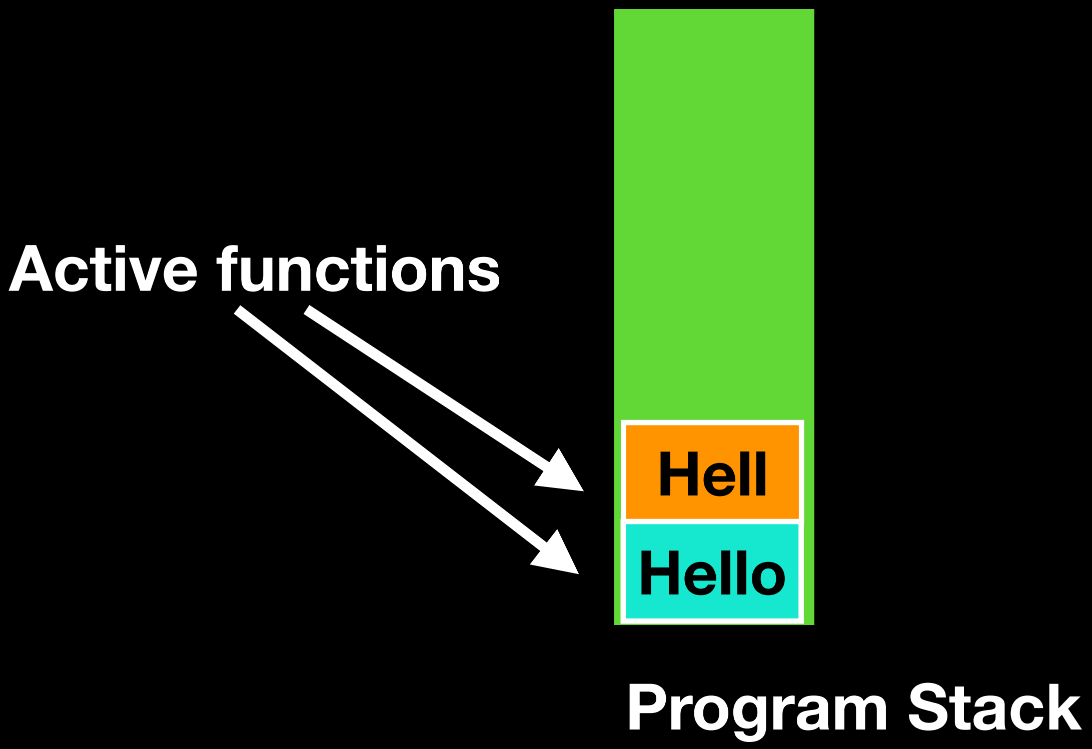
o l l

→ He

o l l e

→ H

o l l e H



Active functions

Hell

Hello

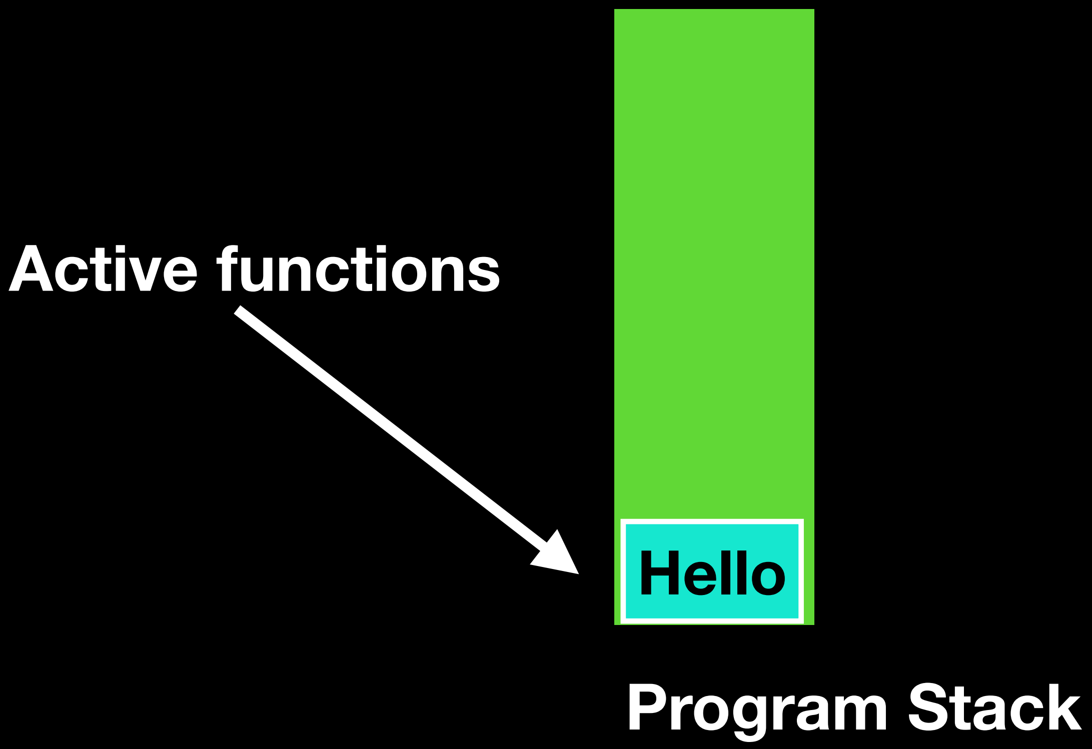
Program Stack

Print String Backwards

```
Hello
o
→   Hell
    o l

→       Hel
        o l l

→           He
            o l l e
→               H
                o l l e H
```



Print String Backwards

Hello

o

→ Hell

o l

→ Hel

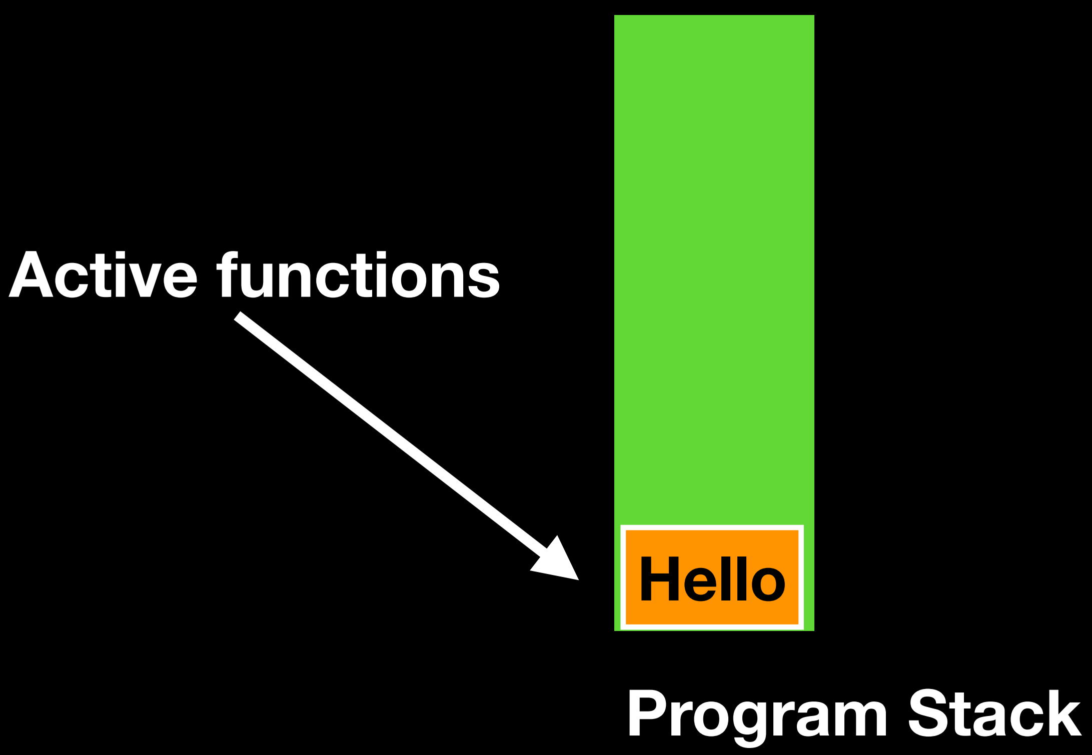
o l l

→ He

o l l e

→ H

o l l e H



Active functions

Hello

Program Stack

Print String Backwards

Hello

o

→ Hell

o l

→ Hel

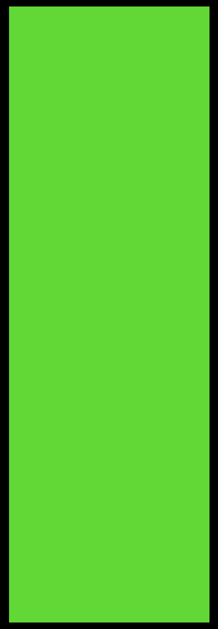
o l l

→ He

o l l e

→ H

o l l e H

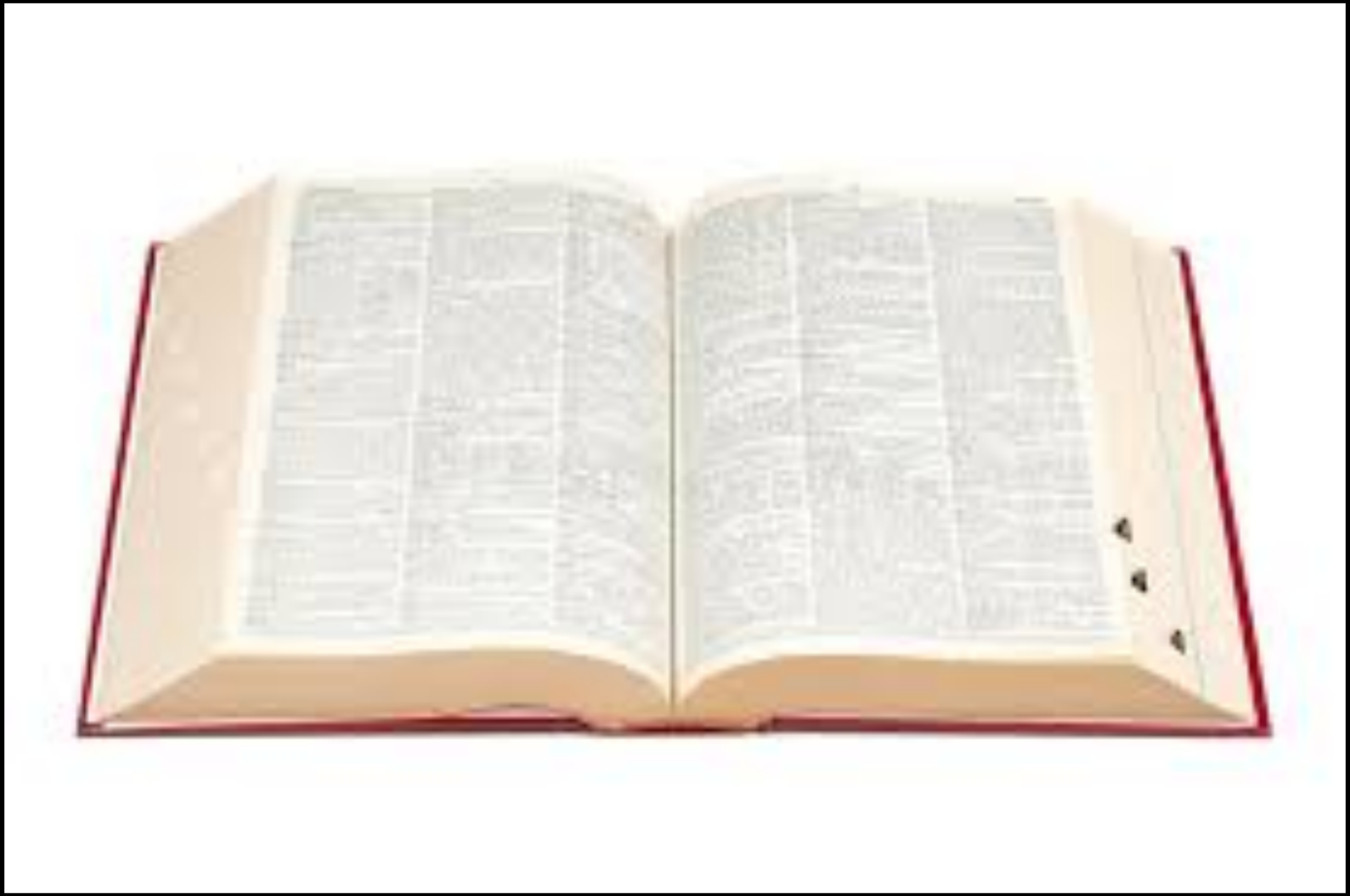


Program Stack

Lecture Activity

If I hand you a **printed** dictionary (an actual book) and ask you to find the word “Kalimba”, what do you do?

Write down precise steps (a procedure) as if someone who doesn’t know what a dictionary is must follow your instructions.



Look in ?

```
LOOK FOR WORD “Kalimba” IN DICTIONARY

- Open dictionary at random page

_ If “Kalimba” is on page FOUND!!!

- Else if “Kalimba” is lexicographically < first word on page
    LOOK FOR WORD “Kalimba” IN LOWER HALF ←
                                            Recursive Call

- Else if “Kalimba” is lexicographically > last word on page
    LOOK FOR WORD “Kalimba” IN UPPER HALF ←
                                            Recursive Call
```

How is this different from recursive solution to print backwards?

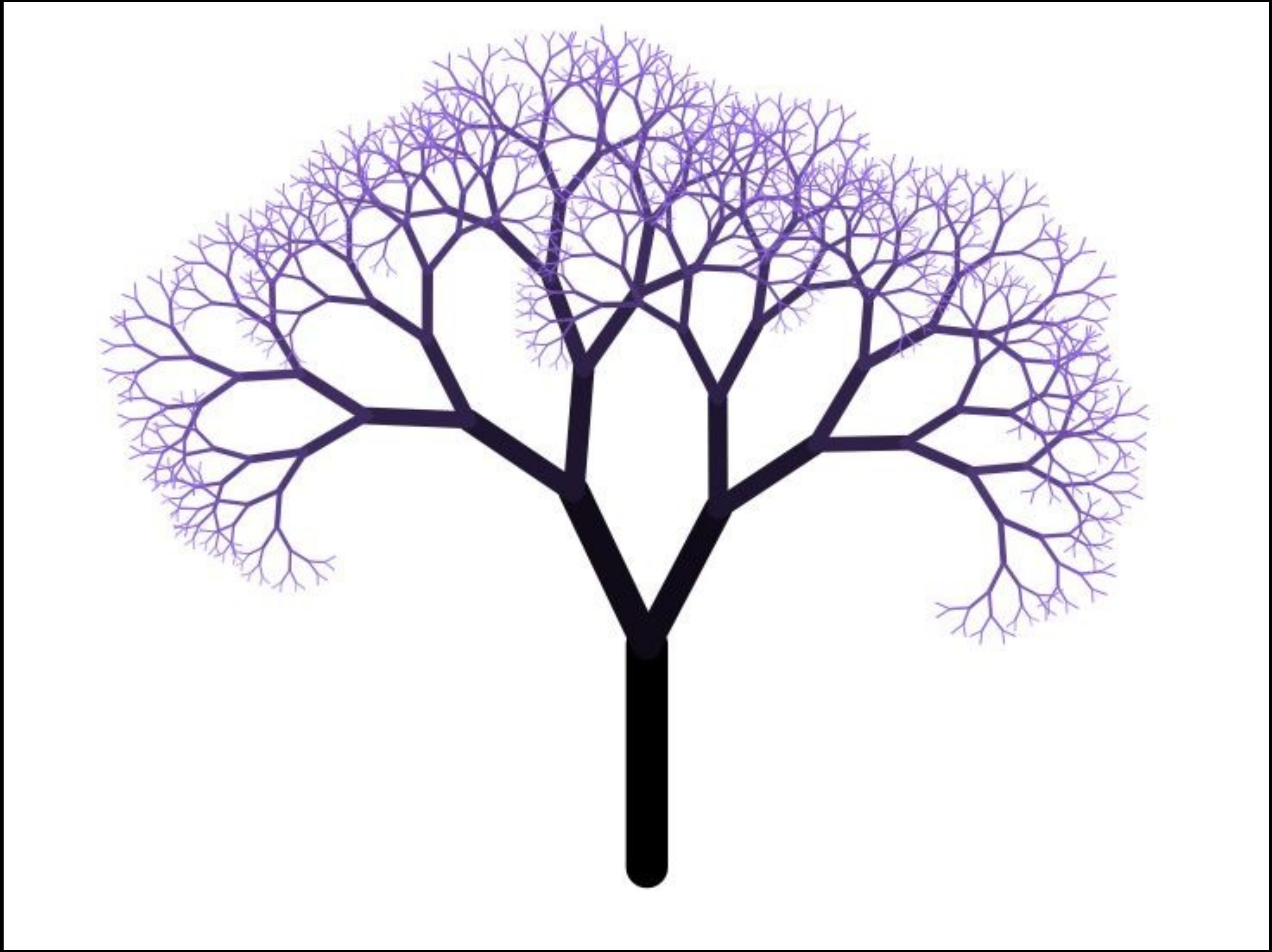
How is this different from recursive solution to print backwards?

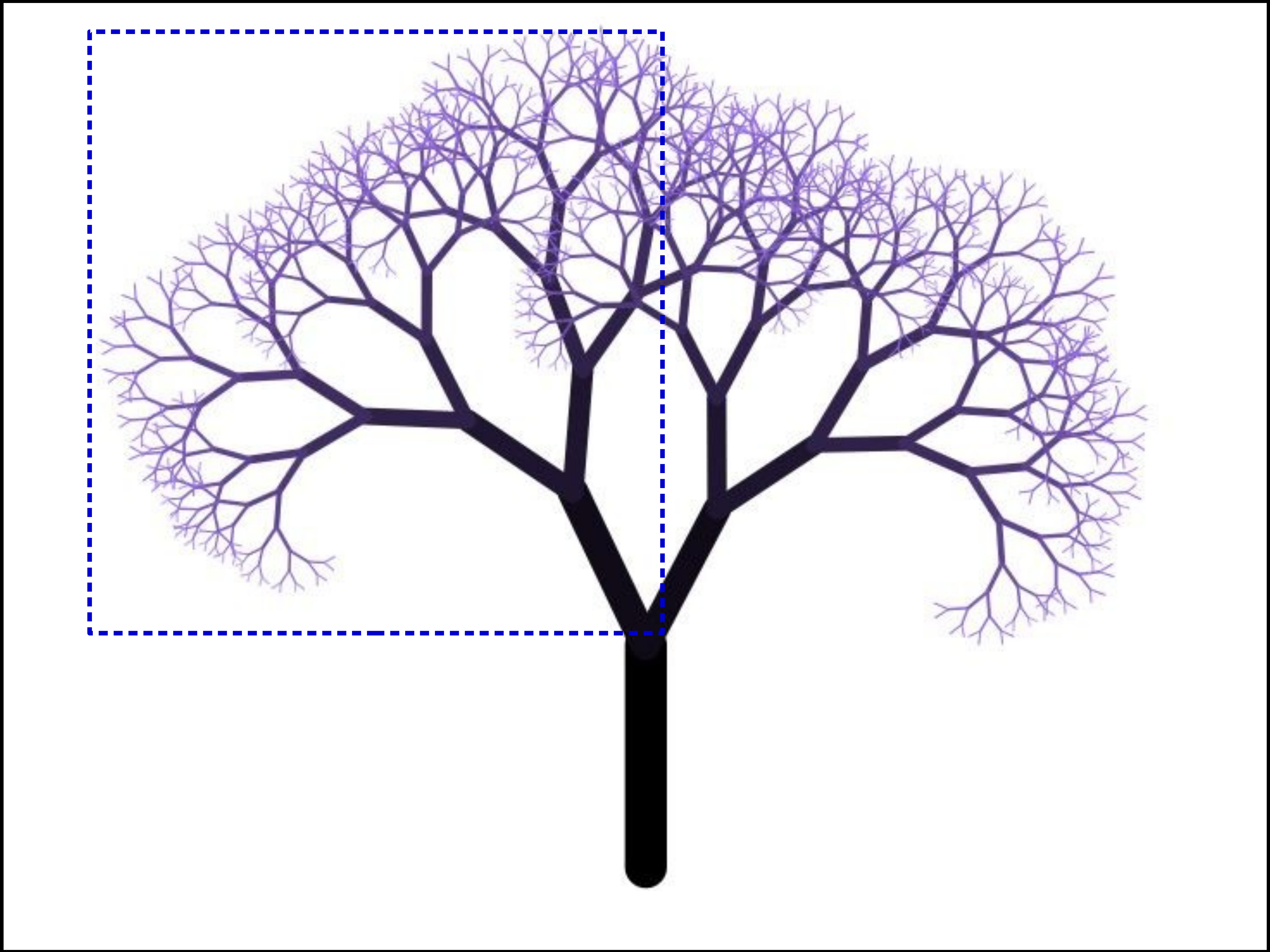
- Two recursive calls

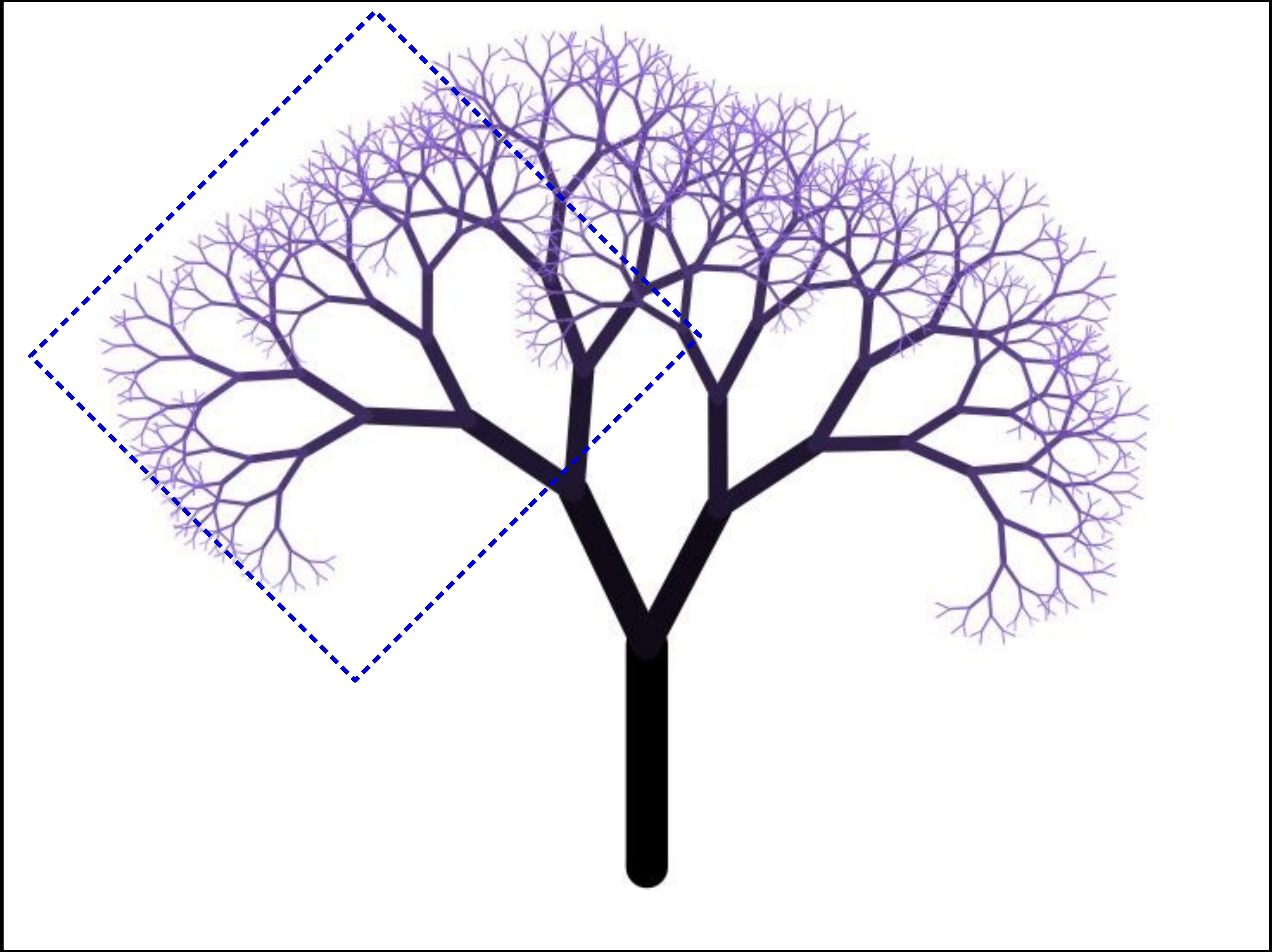
- Execute either one or the other

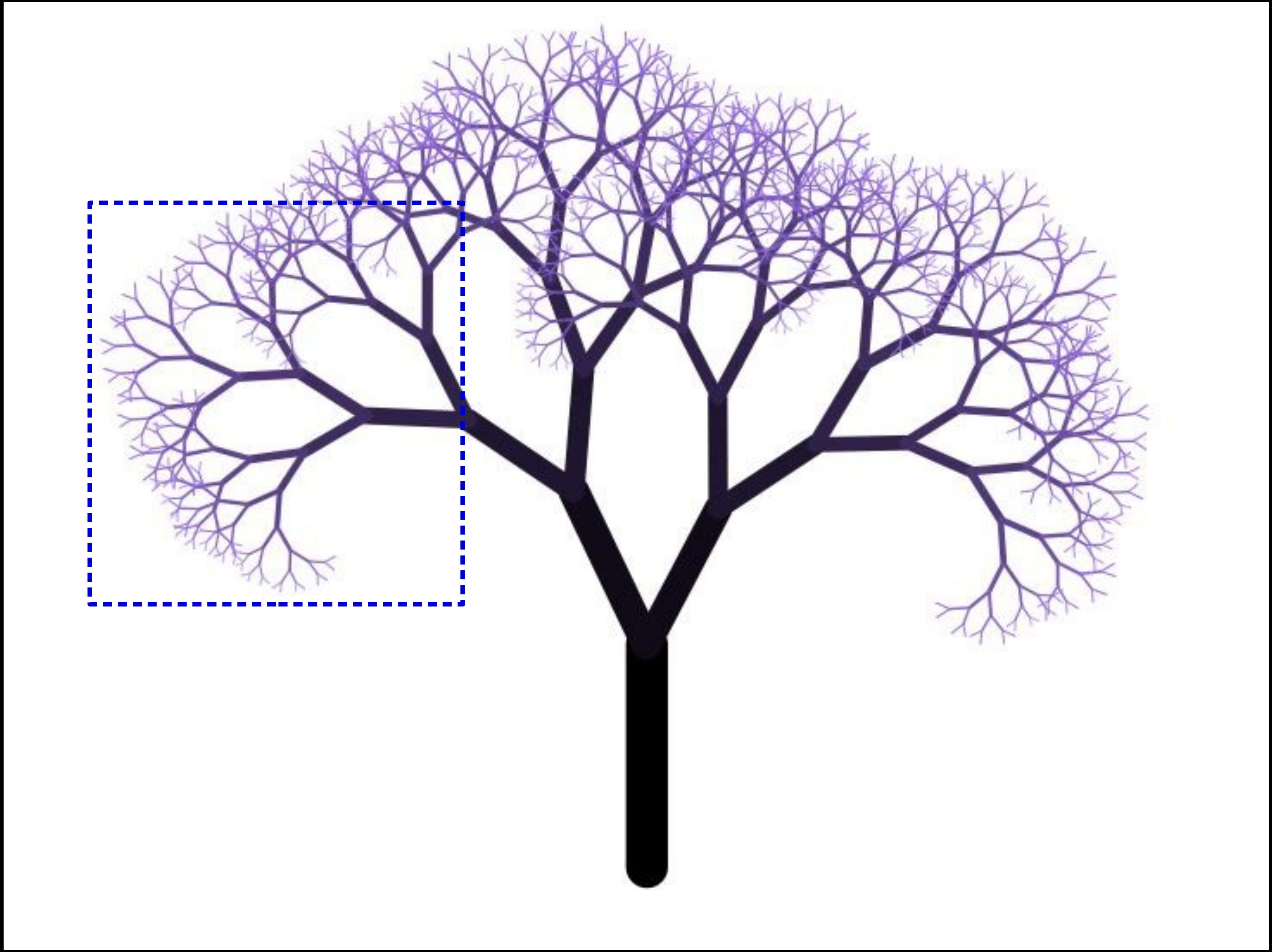
- Cuts problem in 1/2

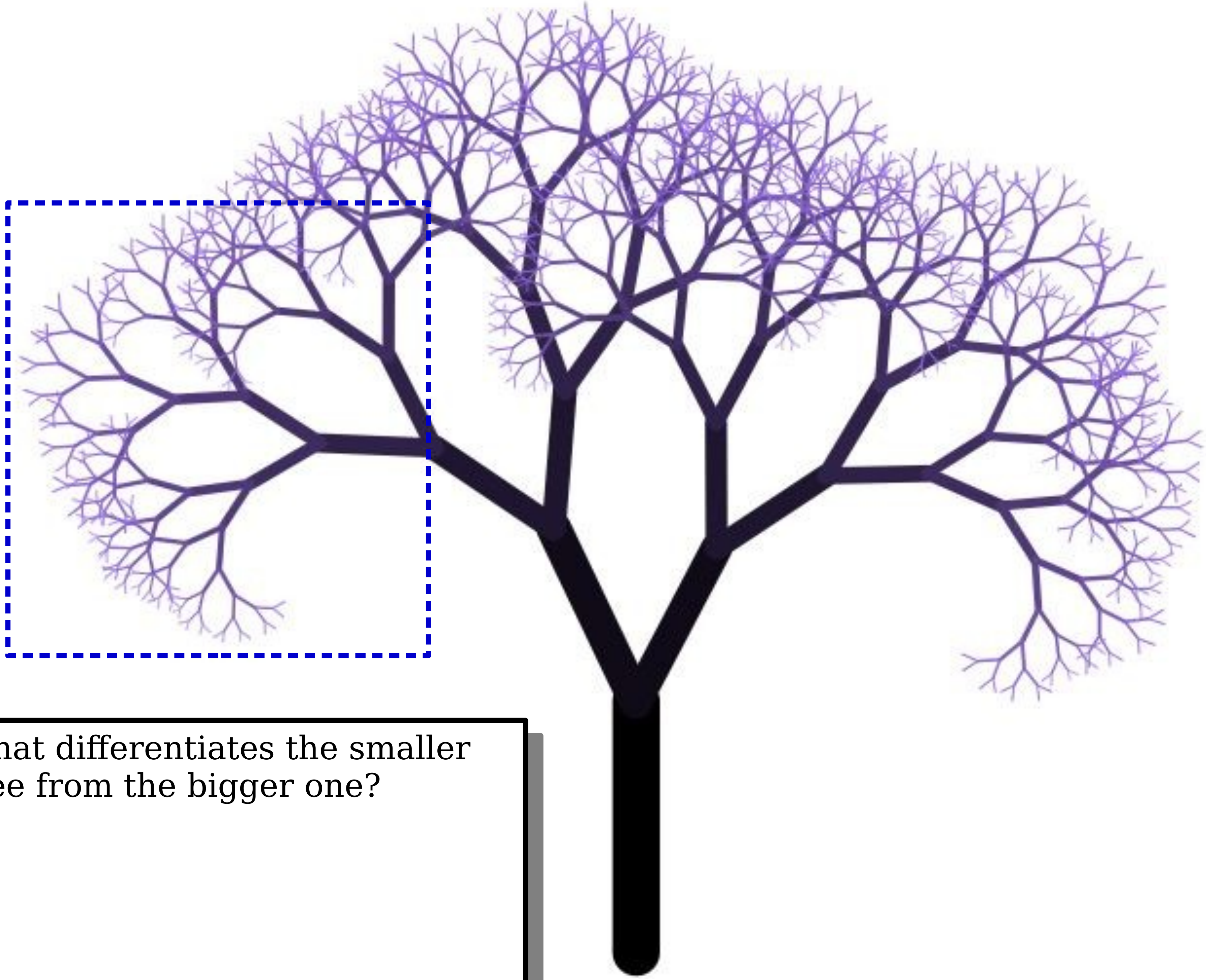
The images in the next slides were adapted from Keith Schwarz at Stanford University



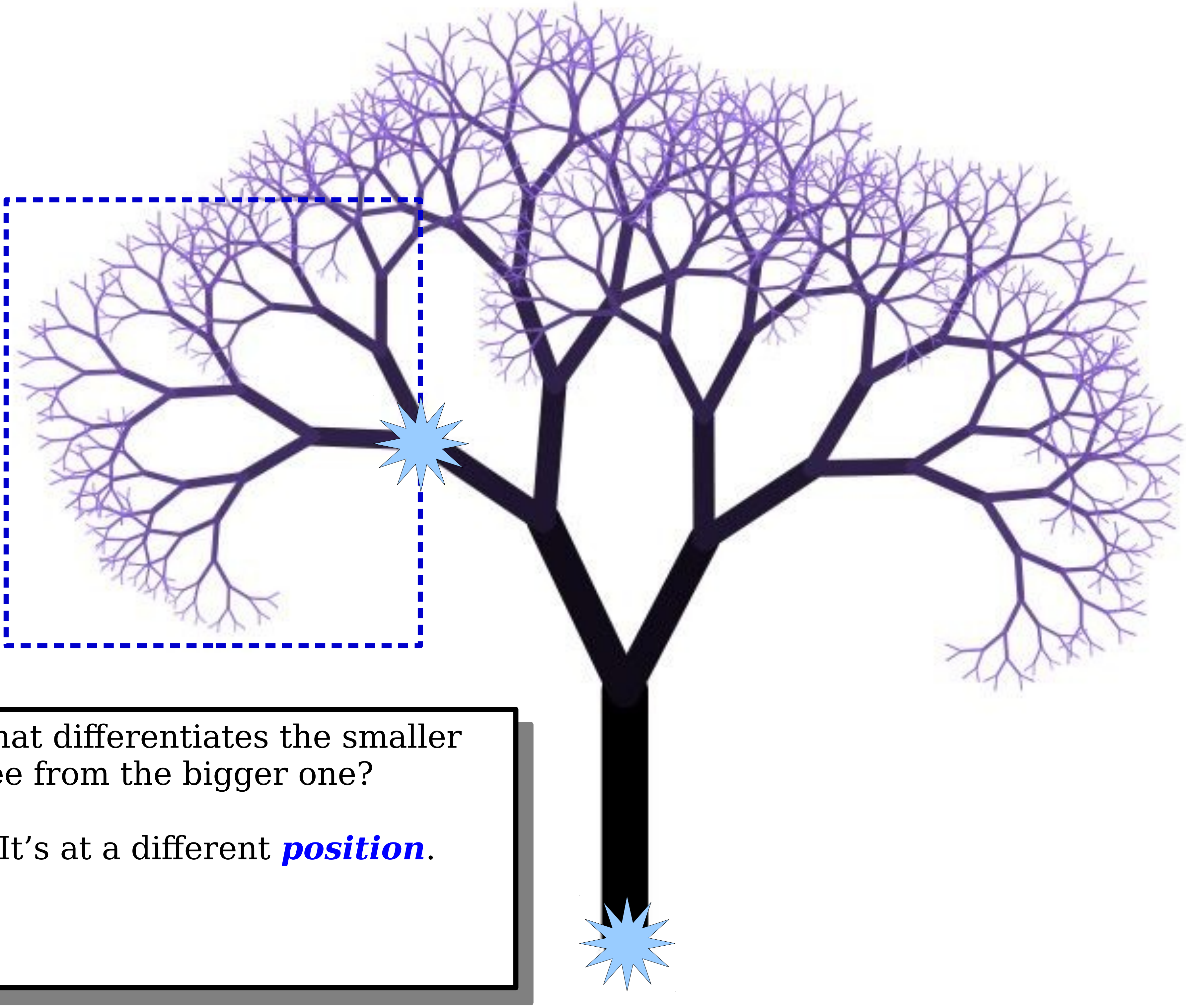






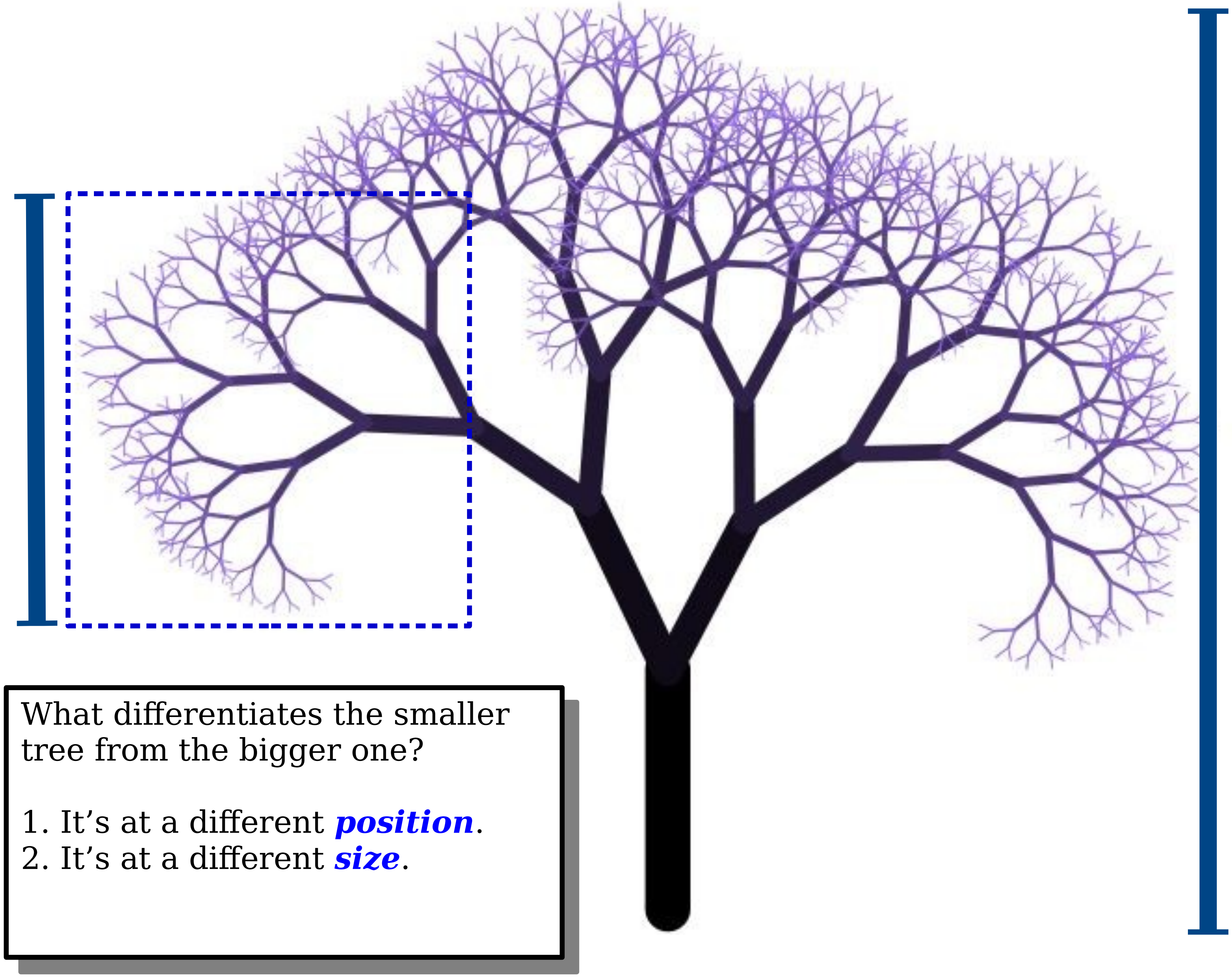


What differentiates the smaller tree from the bigger one?



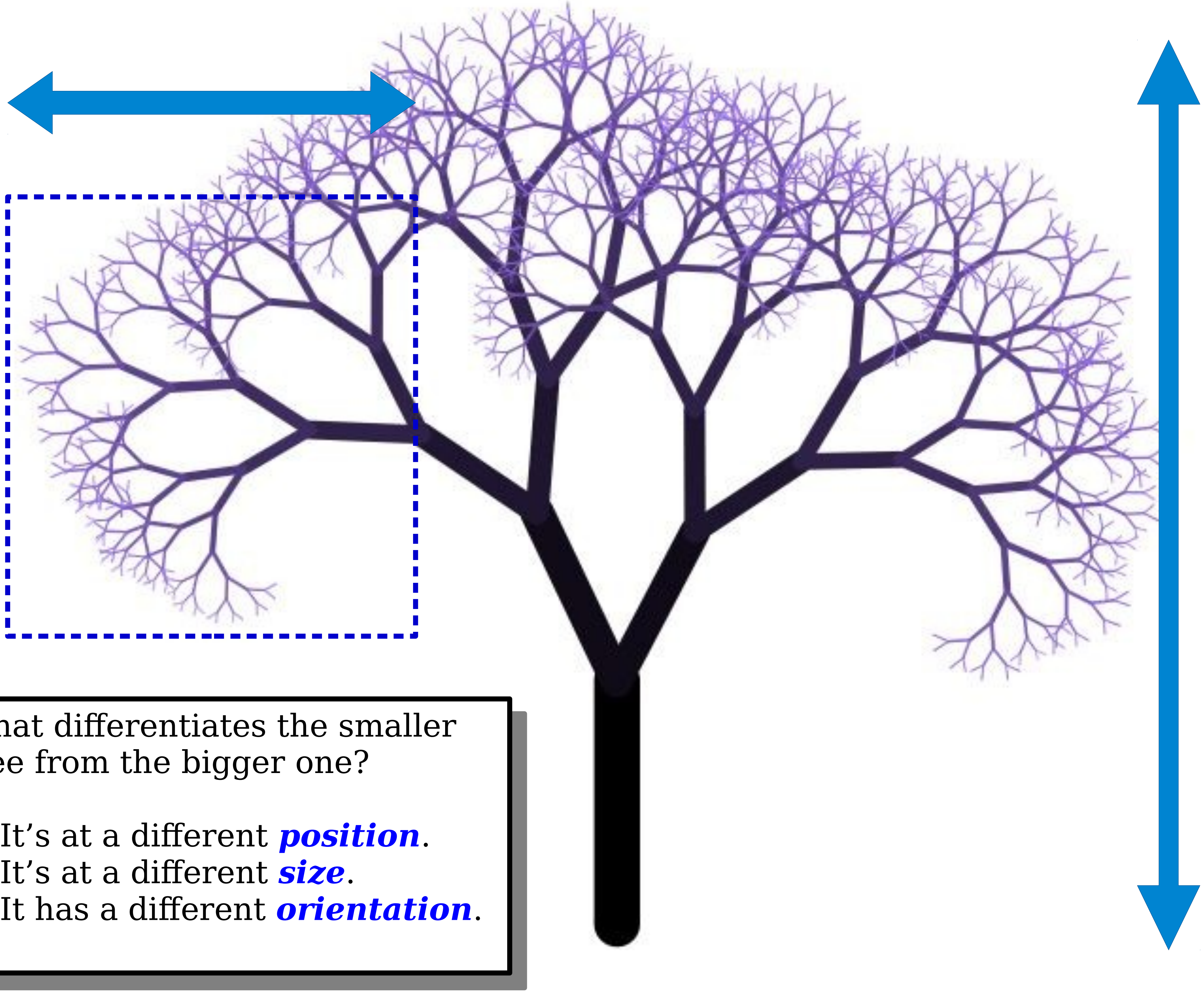
What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.



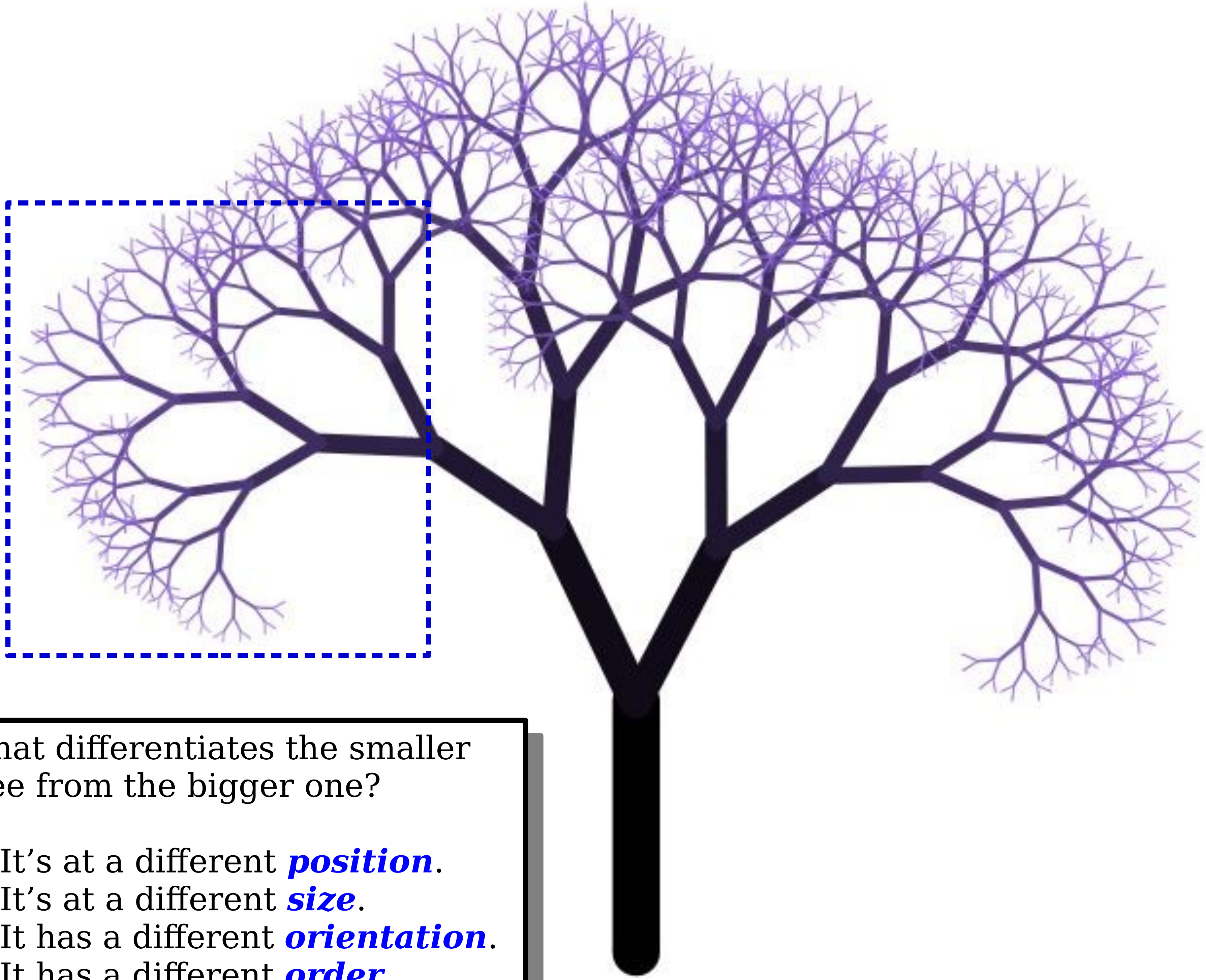
What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.



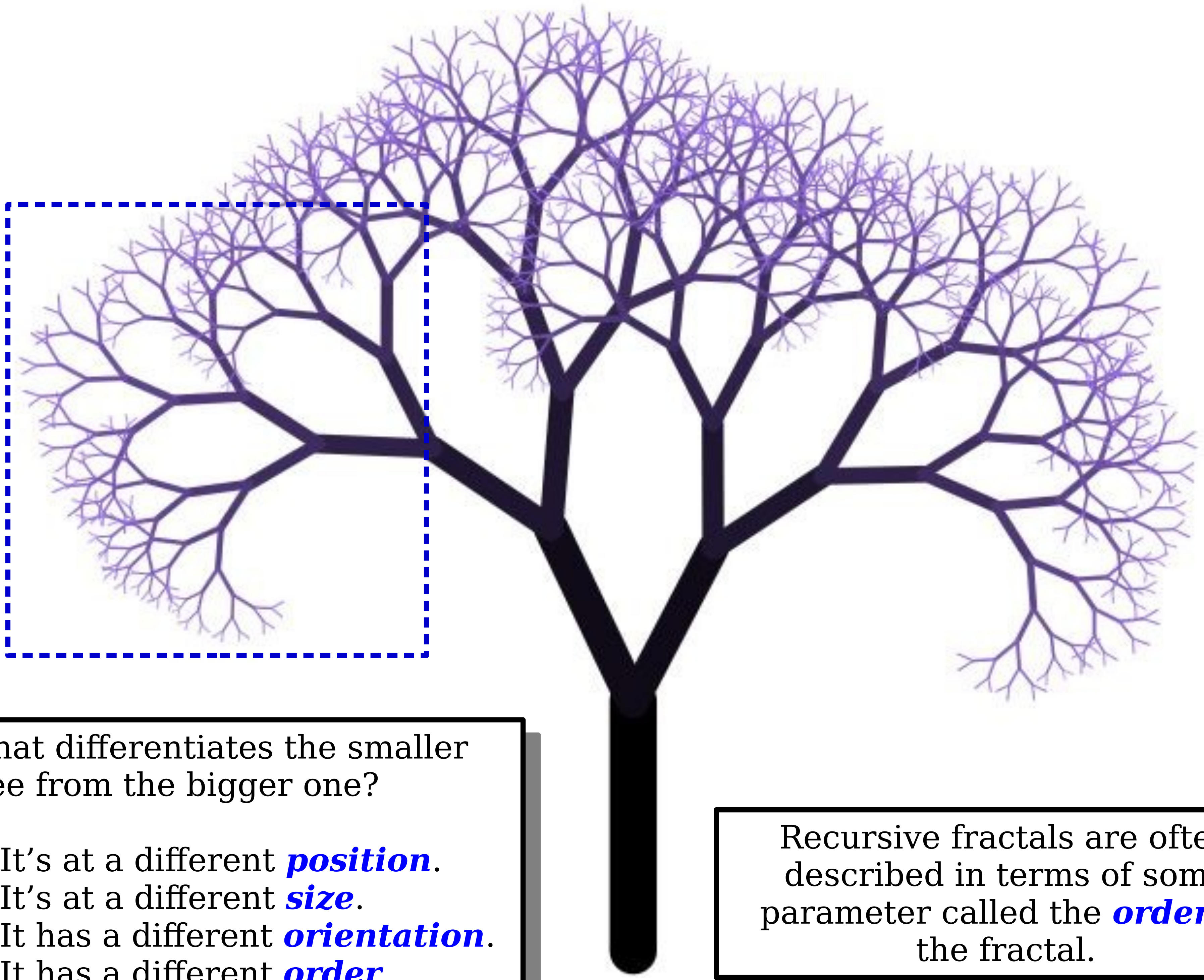
What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.



What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.



What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

An order-0 tree.

What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

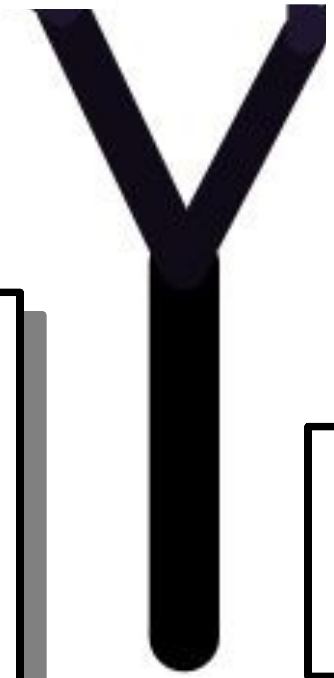
An order-1 tree.

What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

An order-2 tree.

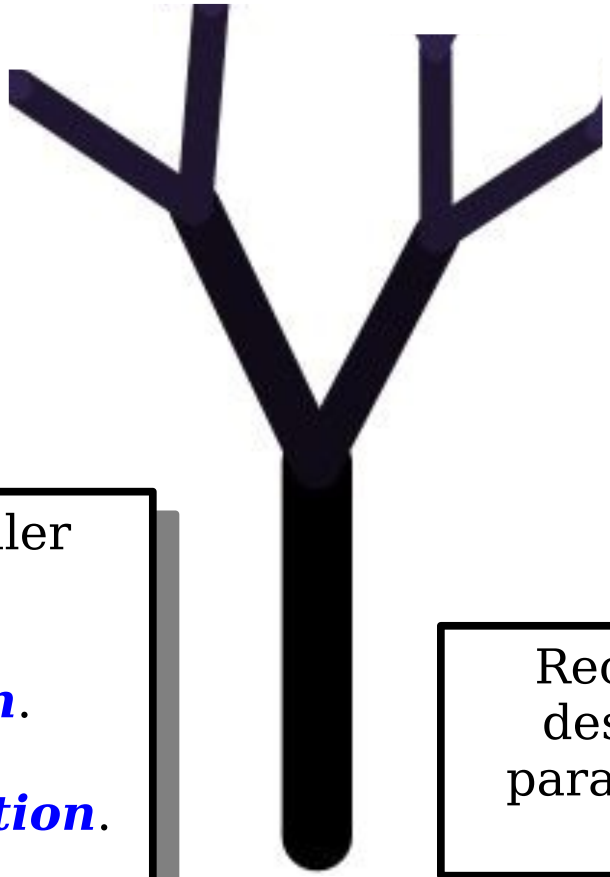


What differentiates the smaller tree from the bigger one?

1. It's at a different ***position***.
2. It's at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

An order-3 tree.

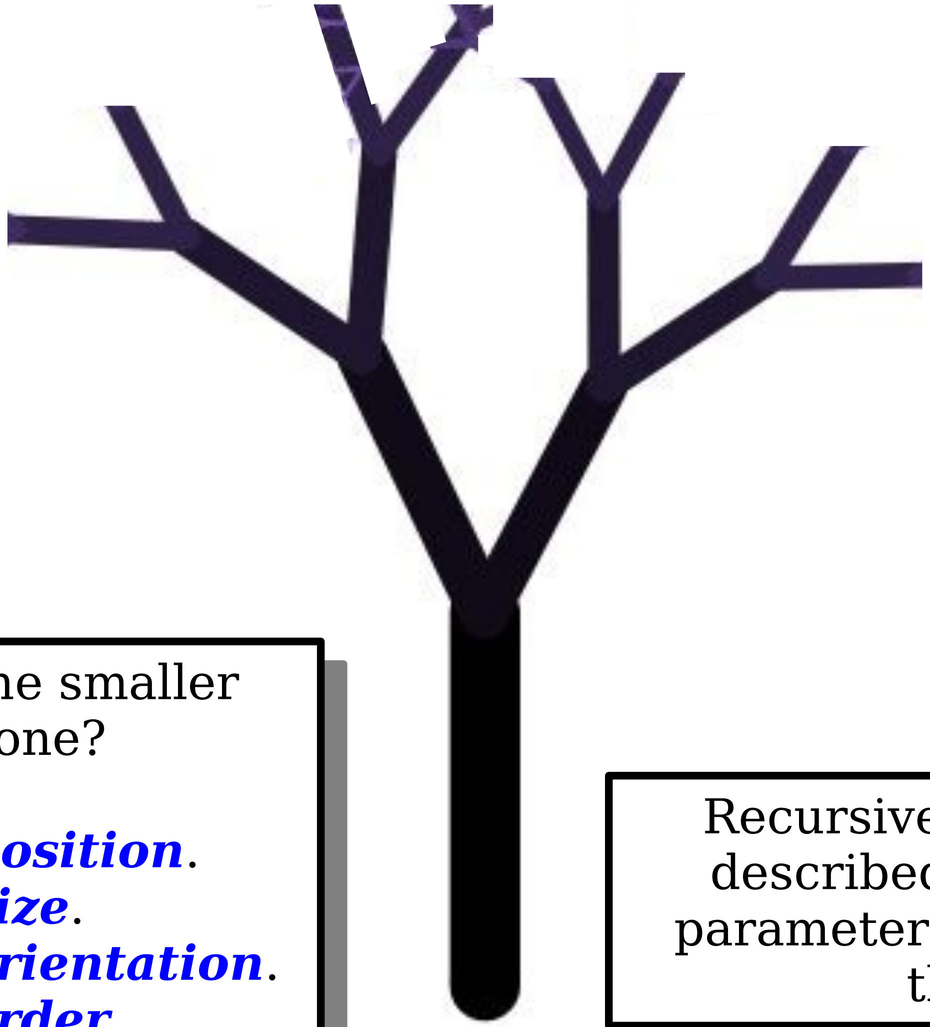


What differentiates the smaller tree from the bigger one?

1. It's at a different ***position***.
2. It's at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

An order-4 tree.

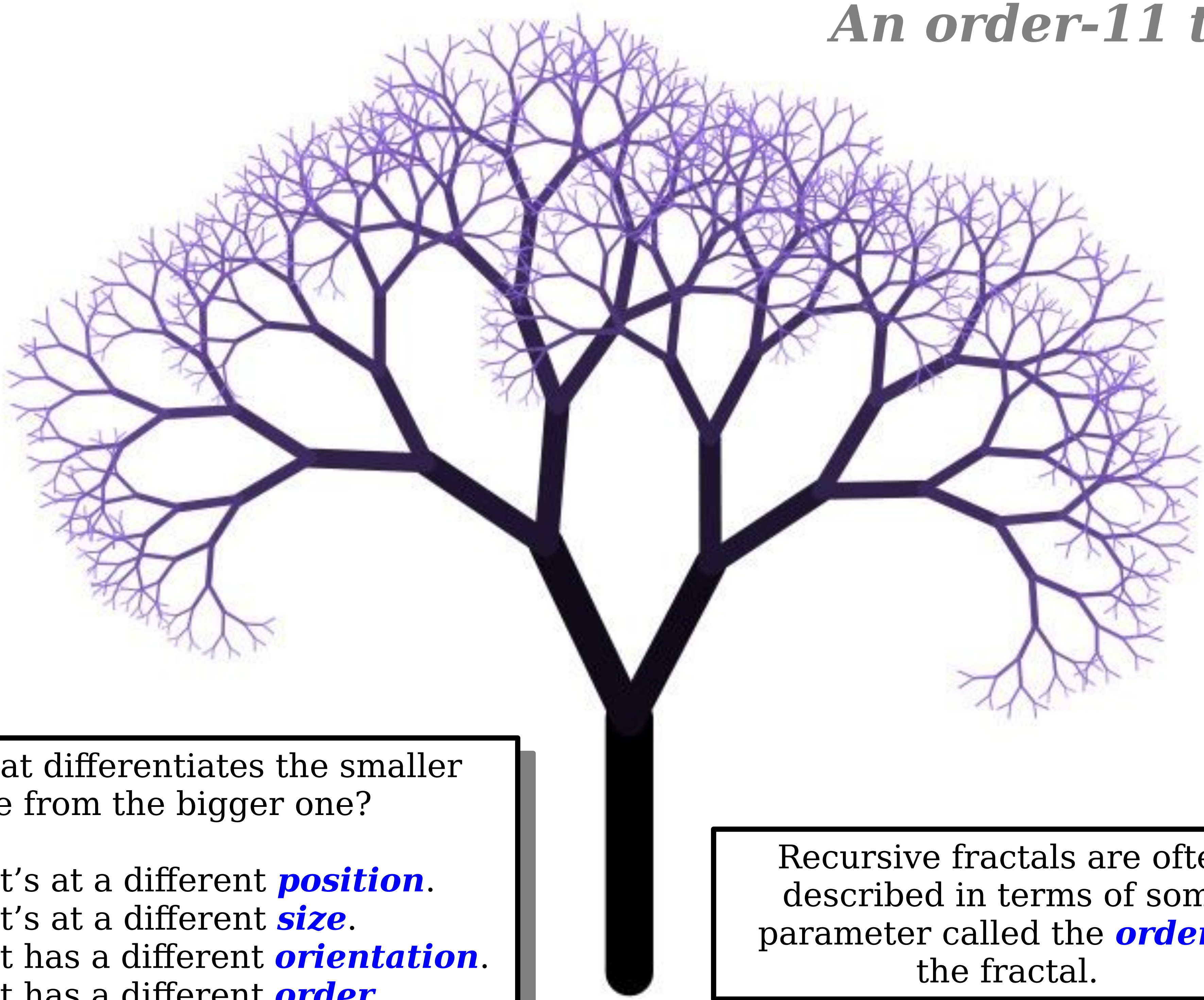


What differentiates the smaller tree from the bigger one?

1. It’s at a different ***position***.
2. It’s at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

An order-11 tree.

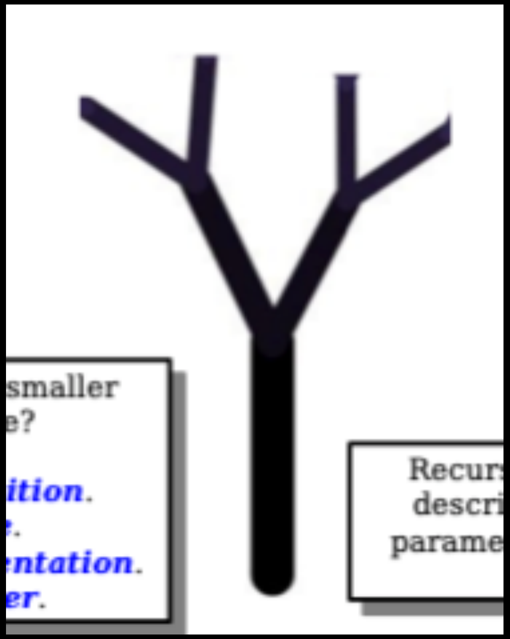


What differentiates the smaller tree from the bigger one?

1. It's at a different ***position***.
2. It's at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

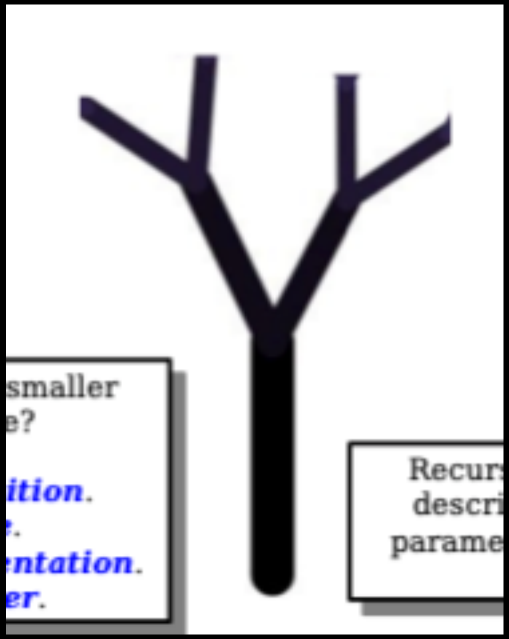
Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

Lecture Activity



Give a sequence of precise instructions in English (algorithm) to DRAW an order-3 fractal tree

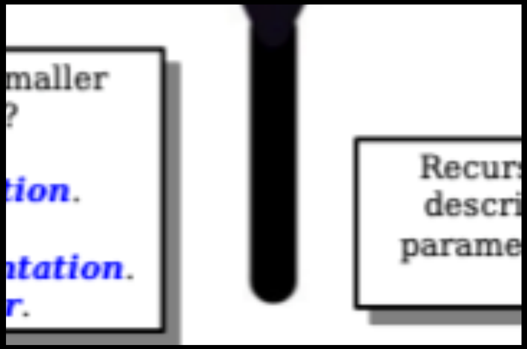
Lecture Activity



Give a sequence of precise instructions in English (algorithm) to DRAW an order-3 fractal tree

Hint:

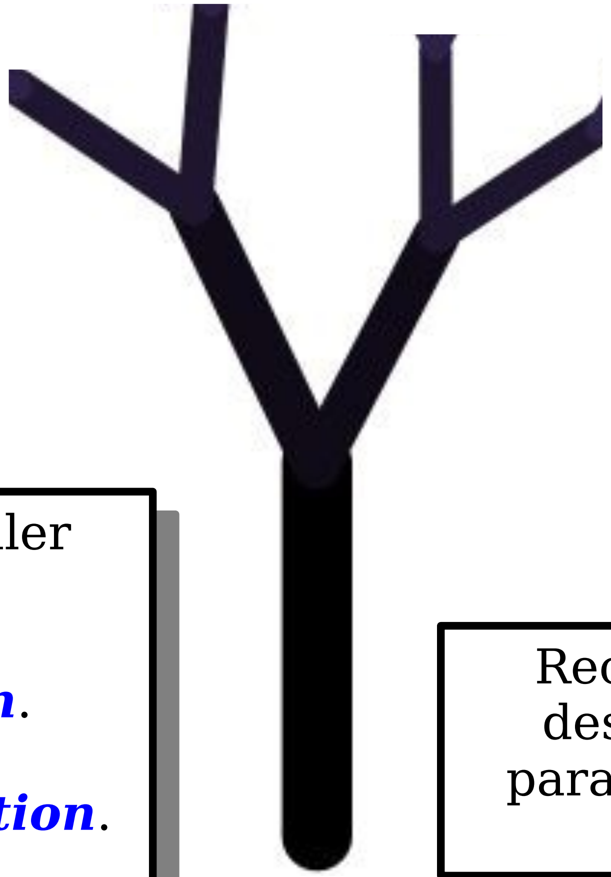
Draw a line and then



An order-3 tree.

An order-0 tree is nothing at all.

An order-*n* tree is a line with two smaller order-(*n*-1) trees starting at the end of that line.



What differentiates the smaller tree from the bigger one?

1. It's at a different ***position***.
2. It's at a different ***size***.
3. It has a different ***orientation***.
4. It has a different ***order***.

Recursive fractals are often described in terms of some parameter called the ***order*** of the fractal.

Lecture Activity

- draw a line

Recursive Call

- tilt the canvas 45˚ left and draw an order-2 tree
- tilt the canvas 45˚ right and draw an order-2 tree

Recursive Call

Lecture Activity

- draw a line

- tilt the canvas 45˚ left and draw an order-2 tree

- tilt the canvas 45˚ right and draw an order-2 tree

Lecture Activity

- draw a line
- tilt the canvas 45° left and draw an order-2 tree
    - draw a line
    - tilt the canvas 45° left and draw an order-1 tree
    - tilt the canvas 45° right and draw an order-1 tree
- tilt the canvas 45° right and draw an order-2 tree
    - draw a line
    - tilt the canvas 45° left and draw an order-1 tree
    - tilt the canvas 45° right and draw an order-1 tree

Lecture Activity

- draw a line
- tilt the canvas 45˚ left and draw an order-2 tree
    - draw a line
    - tilt the canvas 45˚ left and draw an order-1 tree
    - tilt the canvas 45˚ right and draw an order-1 tree
- tilt the canvas 45˚ right and draw an order-2 tree
    - draw a line
    - tilt the canvas 45˚ left and draw an order-1 tree
    - tilt the canvas 45˚ right and draw an order-1 tree

Lecture Activity

- draw a line
- tilt the canvas 45° left and draw an order-2 tree
    - draw a line
    - tilt the canvas 45° left and draw an order-1 tree
        - draw a line
        - tilt the canvas 45° left and draw an order-0 tree
        - tilt the canvas 45° right and draw an order-0 tree
    - tilt the canvas 45° right and draw an order-1 tree
        - draw a line
        - tilt the canvas 45° left and draw an order-0 tree
        - tilt the canvas 45° right and draw an order-0 tree
- tilt the canvas 45° right and draw an order-2 tree
    - draw a line
    - tilt the canvas 45° left and draw an order-1 tree
        - draw a line
        - tilt the canvas 45° left and draw an order-0 tree
        - tilt the canvas 45° right and draw an order-0 tree
    - tilt the canvas 45° right and draw an order-1 tree
        - draw a line
        - tilt the canvas 45° left and draw an order-0 tree
        - tilt the canvas 45° right and draw an order-0 tree

Lecture Activity

- draw a line
- tilt the canvas 45˚ left and draw an order-2 tree
    - draw a line
    - tilt the canvas 45˚ left and draw an order-1 tree
        - draw a line
        - tilt the canvas 45˚ left and draw an order-0 tree
        - tilt the canvas 45˚ right and draw an order-0 tree
    - tilt the canvas 45˚ right and draw an order-1 tree
        - draw a line
        - tilt the canvas 45˚ left and draw an order-0 tree
        - tilt the canvas 45˚ right and draw an order-0 tree
- tilt the canvas 45˚ right and draw an order-2 tree
    - draw a line
    - tilt the canvas 45˚ left and draw an order-1 tree
        - draw a line
        - tilt the canvas 45˚ left and draw an order-0 tree
        - tilt the canvas 45˚ right and draw an order-0 tree
    - tilt the canvas 45˚ right and draw an order-1 tree
        - draw a line
        - tilt the canvas 45˚ left and draw an order-0 tree
        - tilt the canvas 45˚ right and draw an order-0 tree

Nothing to draw at order 0
We stop!

BASE CASE

In general for n

- draw a line

- tilt the canvas 45° left and draw and order-(n-1) tree

- tilt the canvas 45° right and draw and order-(n-1) tree

Check This Out!!!

http://recursivedrawing.com/

Different Flavors of Recursion

Reverse String: write first character, reverse the remaining single smaller string

Dictionary: either inspect upper-half or lower-half

Fractal Tree: draw both the left order-(n-1) and right order-(n-1) trees

All solve a problem by breaking it up into one or more smaller “*similar*” problems

Recursive Problem-Solving

```
if (problem is sufficiently simple) {

    directly solve the problem
    i.e. do something and/or return the solution

} else {

    split problem up into one or more smaller problems with the same structure as the original

    solve some or all of those smaller problems

    do something or combine results to return solution if necessary
}
```

Recursive Problem-Solving

```
if(problem is sufficiently simple){

     directly solve the problem
     i.e. do something and/or return the solution

} else{

     split problem up into one or more smaller
  problems with the same structure as the original

     solve some or all of those smaller problems

     do something or combine results to return
  solution if necessary
}
```

BASE CASE

Why Recursion

An alternative to iteration

Not always practical (some compilers optimize tail-recursive algorithms)

Elegant and intuitive solution for some problems

Factorial

1 x 2 x 3 x … x n

$$n! = \prod_{k=1}^{n} k$$

For example:

0!=1,**1!**=1, **2!**=2, **3!**=6, **4!**=24, **5!**=120

The empty product

But what if we start from `n`?

n!=

But what if we start from `n`?

$$\mathbf{n!} = n \times \underline{(n-1) \times (n-2) \times (n - 3) \times \ldots \quad \ldots. \quad \ldots. \quad \ldots\ 2 \times 1}$$

What is this?

But what if we start from `n`?

$$n! = n \times \underbrace{(n-1) \times (n-2) \times (n - 3) \times \ldots \quad \ldots. \quad \ldots. \quad \ldots\, 2 \times 1}_{(n-1)!}$$

But what if we start from `n`?

$$n! = n \text{ x } \underbrace{(n-1) \text{ x } (n-2) \text{ x } (n - 3) \text{ x } \ldots \quad \ldots. \quad \ldots. \quad \ldots\ 2 \text{ x } 1}_{(n-1)!}$$

$$(n-1)! = (n-1) \text{ x } \underbrace{(n-2) \text{ x } (n - 3) \text{ x } \ldots \quad \ldots. \quad \ldots. \quad \ldots\ 2 \text{ x } 1}_{\text{What is this?}}$$

But what if we start from `n`?

$$n! = n \text{ x } \underbrace{(n-1) \text{ x } (n-2) \text{ x } (n - 3) \text{ x } \ldots \quad \ldots. \quad \ldots. \quad \ldots\, 2 \text{ x } 1}_{(n-1)!}$$

$$(n-1)! = (n-1) \text{ x } \underbrace{(n-2) \text{ x } (n - 3) \text{ x } \ldots \quad \ldots. \quad \ldots. \quad \ldots\, 2 \text{ x } 1}_{(n-2)!}$$

Recursion that Returns a Value

$$n! = n \times (n-1)!$$

Same function being called within solution

Recursion that Returns a Value

n! = n x (n-1)!

```
/** Computes the factorial of the nonnegative integer n.
 @pre:  n must be greater than or equal to 0.
 @post:  None.
 @return:  The factorial of n; n is unchanged. */
int factorial(int n)
{
   if (n == 0)
      return 1;
   else // n > 0 : n-1 >= 0, fact(n-1) returns (n-1)!
      return n * factorial(n - 1); // n * (n-1)! is n!
}  // end fact
```

Recursion that Returns a Value

n! = n x (n-1)!

```
/** Computes the factorial of the nonnegative integer n.
 @pre:  n must be greater than or equal to 0.
 @post:  None.
 @return:  The factorial of n; n is unchanged. */
int factorial(int n)
{
   if (n == 0)
      return 1;
   else // n > 0 : n-1 >= 0, fact(n-1) returns (n-1)!
      return n * factorial(n - 1); // n * (n-1)! is n!
}  // end fact
```

BASE CASE

Recursion that Returns a Value

n! = n x (n-1)!

```
/** Computes the factorial of the nonnegative integer n.
 @pre:  n must be greater than or equal to 0.
 @post:  None.
 @return:  The factorial of n; n is unchanged. */
int factorial(int n)
{
   if (n == 0)
      return 1;
   else // n > 0 : n-1 >= 0, fact(n-1) returns (n-1)!
      return n * factorial(n - 1); // n * (n-1)! is n!
}  // end fact
```

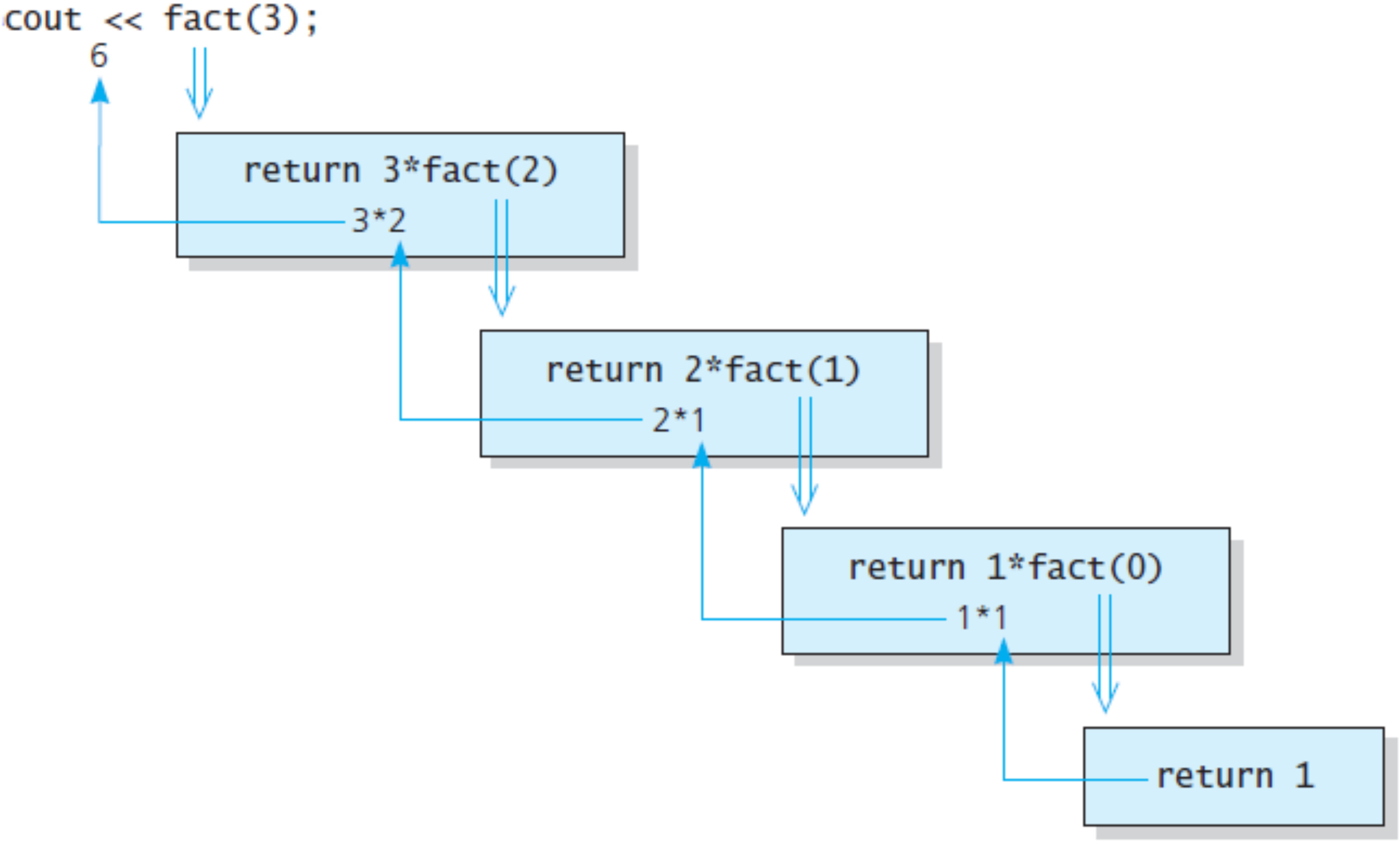
Recursion that Returns a Value

n! = n x (n-1)!

```
/** Computes the factorial of the nonnegative integer n.
 @pre:  n must be greater than or equal to 0.
 @post:  None.
 @return:  The factorial of n; n is unchanged. */
int factorial(int n)
{
   if (n == 0)
      return 1;
   else // n > 0 : n-1 >= 0, fact(n-1) returns (n-1)!
      return n * factorial(n - 1); // n * (n-1)! is n!
}  // end fact
```

BASE CASE

WILL LEAD TO BASE CASE



```
cout << fact(3);
6
return 3*fact(2)
3*2
return 2*fact(1)
2*1
return 1*fact(0)
1*1
return 1
```

Writing a String Backwards

```
writeBackward(string s)
{
   if(the string is empty)
      Do nothing - this is the base case
   else
      Write the last character of s
      writeBackward(s minus the last char)
}
```

Recursion that Performs an Action

```
/** Prints a string backward.
  @post:  The string s is printed backwards
 @param: s  The string to write backwards */

void writeBackward(string s)
{
   size_t length = s.size(); // Length of string
   if (length > 0)//implicit base case: if length == 0 do nothing
   {
      // Print the last character
      cout << s.substr(length - 1, 1);

      // Print the rest of the string backwards - recursive call
      writeBackward(s.substr(0, length - 1));
   } // end if
     // length == 0 is the base case - do nothing
} // end writeBackward
```

Recursion that Performs an Action

```
/** Prints a string backward.
  @post:  The string s is printed backwards
 @param: s  The string to write backwards */

void writeBackward(string s)
{
   size_t length = s.size(); // Length of string
   if (length > 0)//implicit base case: if length == 0 do nothing
   {
      // Print the last character
      cout << s.substr(length - 1, 1);

      // Print the rest of the string backwards - recursive call
      writeBackward(s.substr(0, length - 1));
   } // end if
     // length == 0 is the base case - do nothing
} // end writeBackward
```

WILL LEAD TO BASE CASE

Write String Backwards

Hello

o

Hell

o l

Hel

o l l

He

o l l e

H

o l l e H

BASE CASE