

More Recursion

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Today's Plan



Recursion Review

8 Queens Problem

Permutations

Combinations

Types of Recursion

Reverse String:

- single recursive call
- Base case: stop => no return value

Dictionary:

- split problem into halves but solve only 1
- Base case: stop => no return value

Fractal Tree:

- split problem into halves and solve both
- Base case: stop => no return value

Factorial:

- single recursive call
- Base case: return a value for computation in each recursive call

Why/When use recursion

Usually less efficient than iterative counterparts (we will see example later in the course)

Inherent overhead associated with function calls

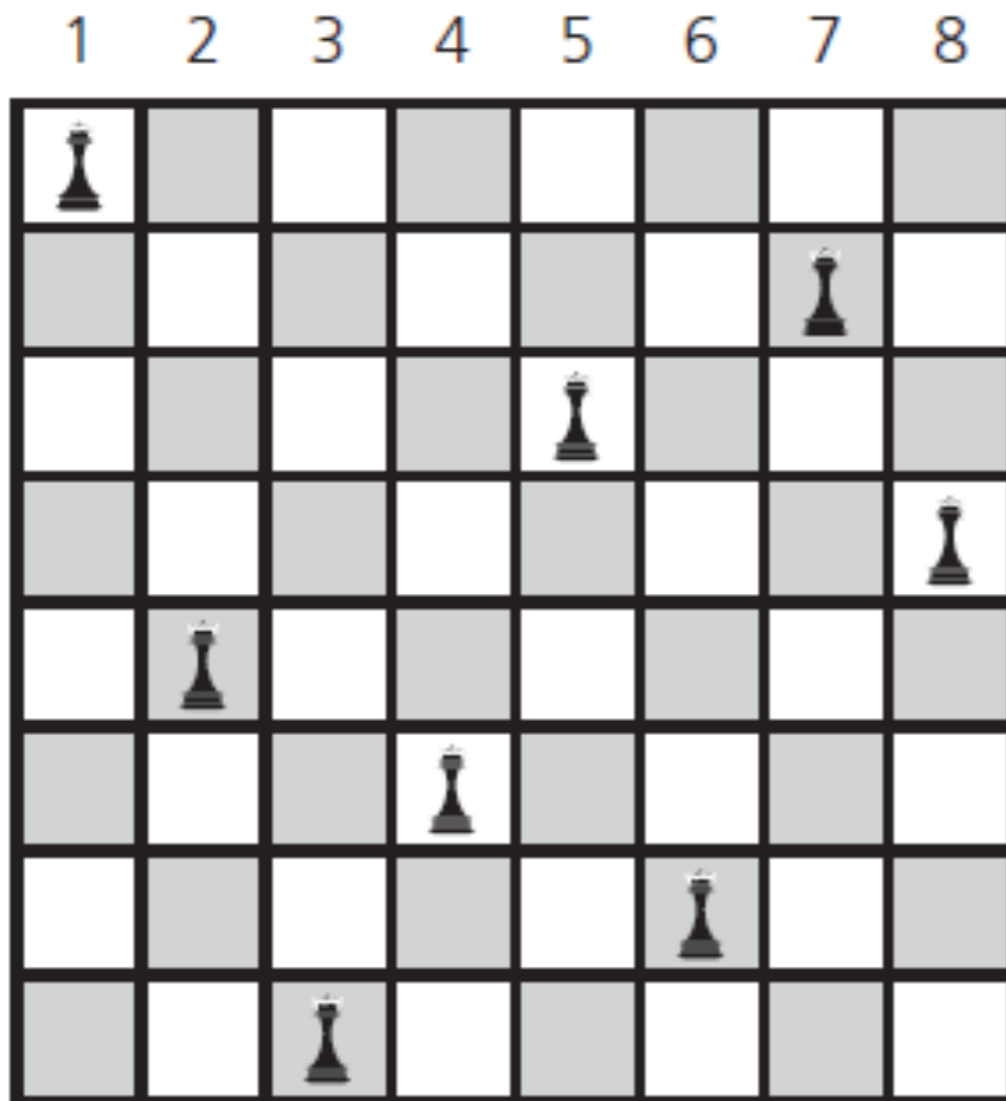
Repeated recursive calls with same parameters

Compilers can optimize tail-recursive (recursive call is the last statement in the function) functions to be iterative

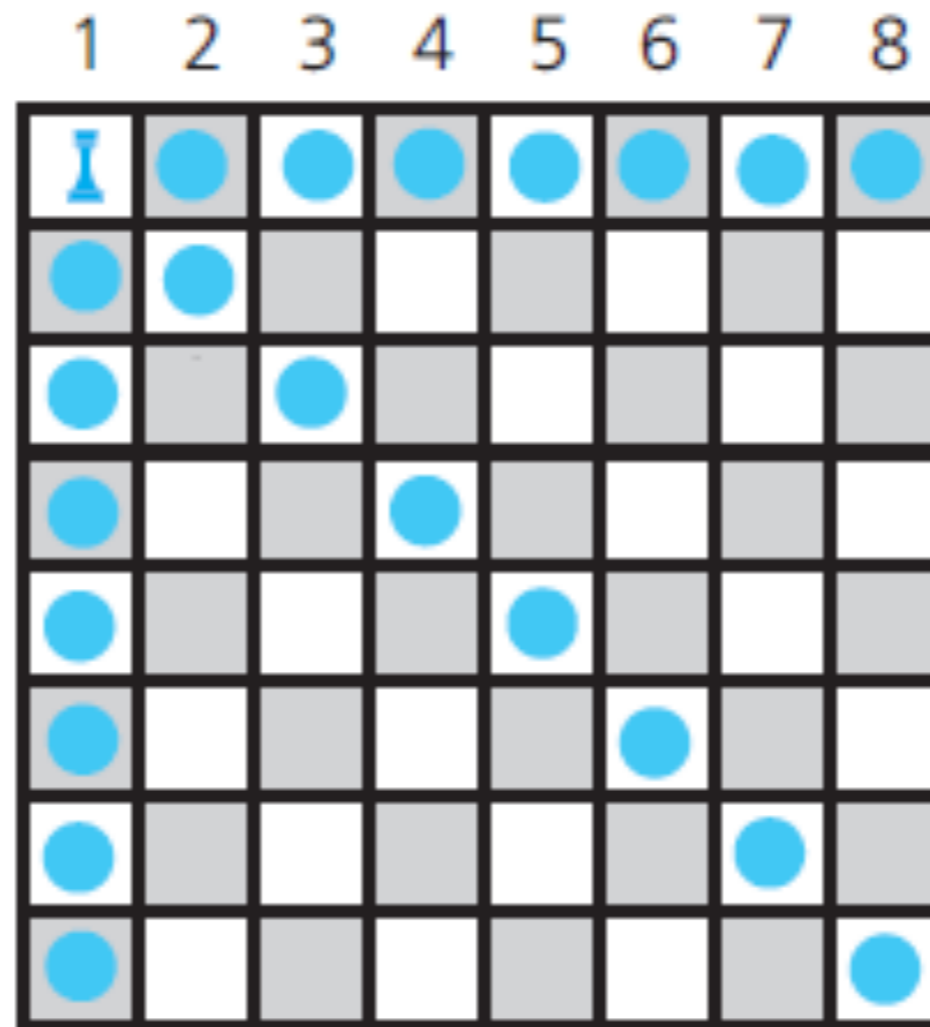
Sometimes logic of iterative solution can be very complex in comparison to recursive solution

The Eight Queens Problem

Place 8 Queens on the board s.t. no queen is on the same row, column or diagonal

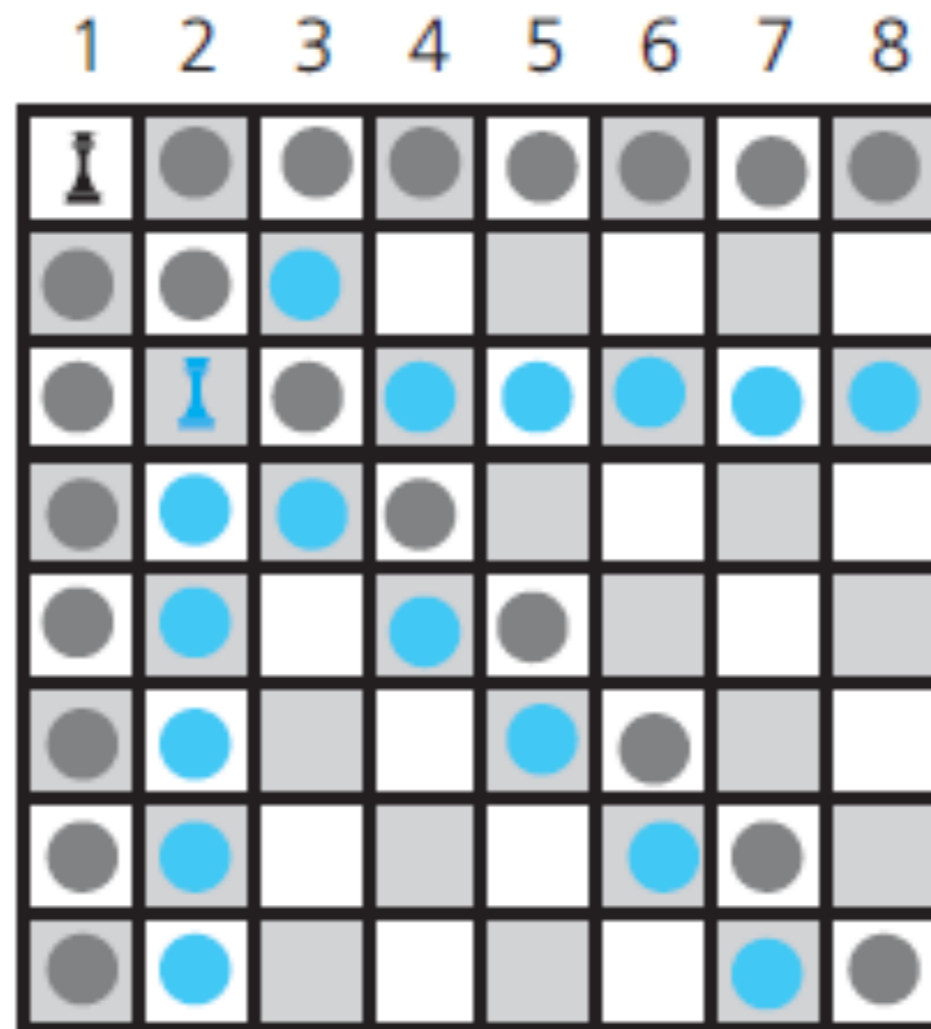


The Eight Queens Problem



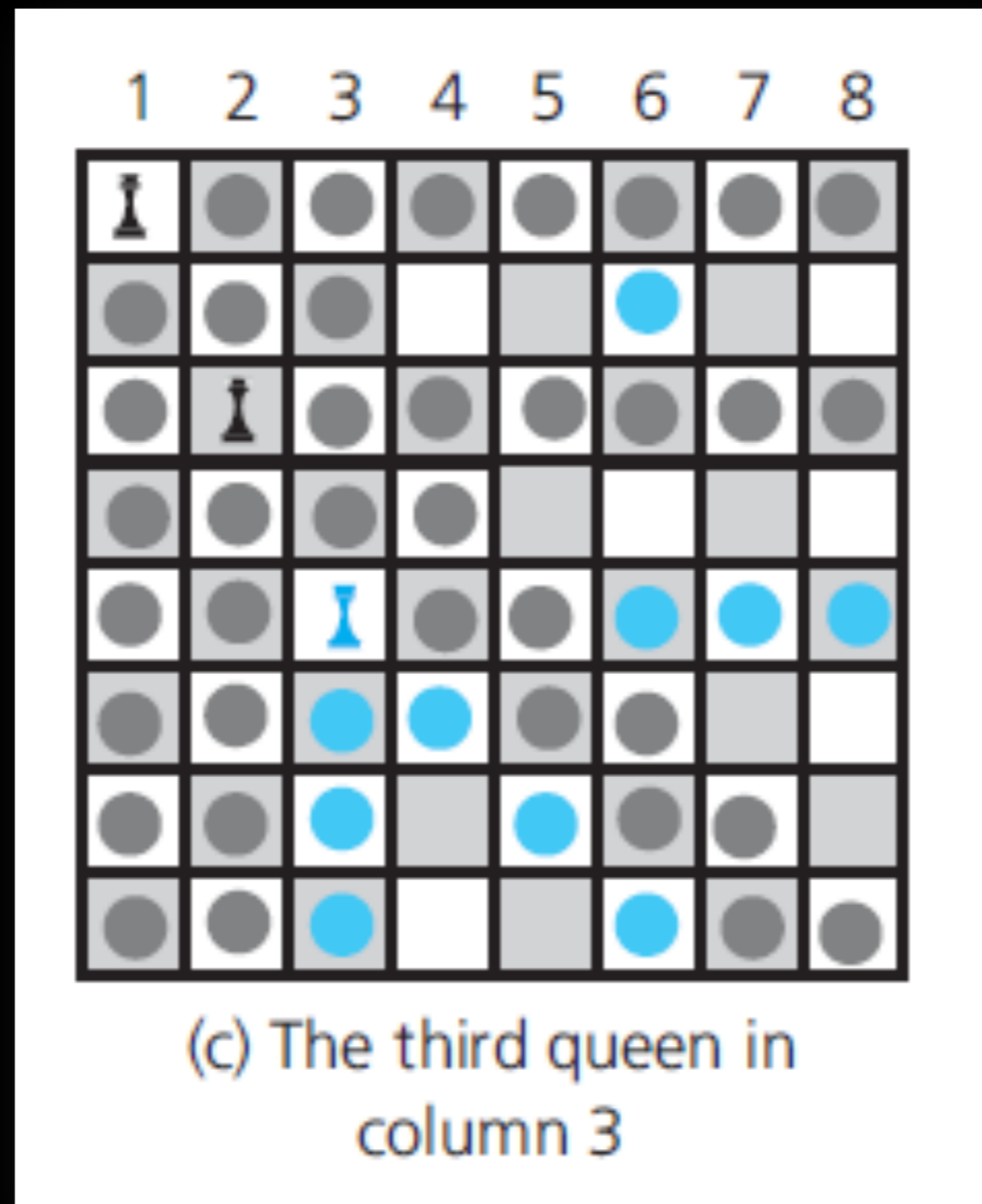
(a) The first queen in column 1

The Eight Queens Problem

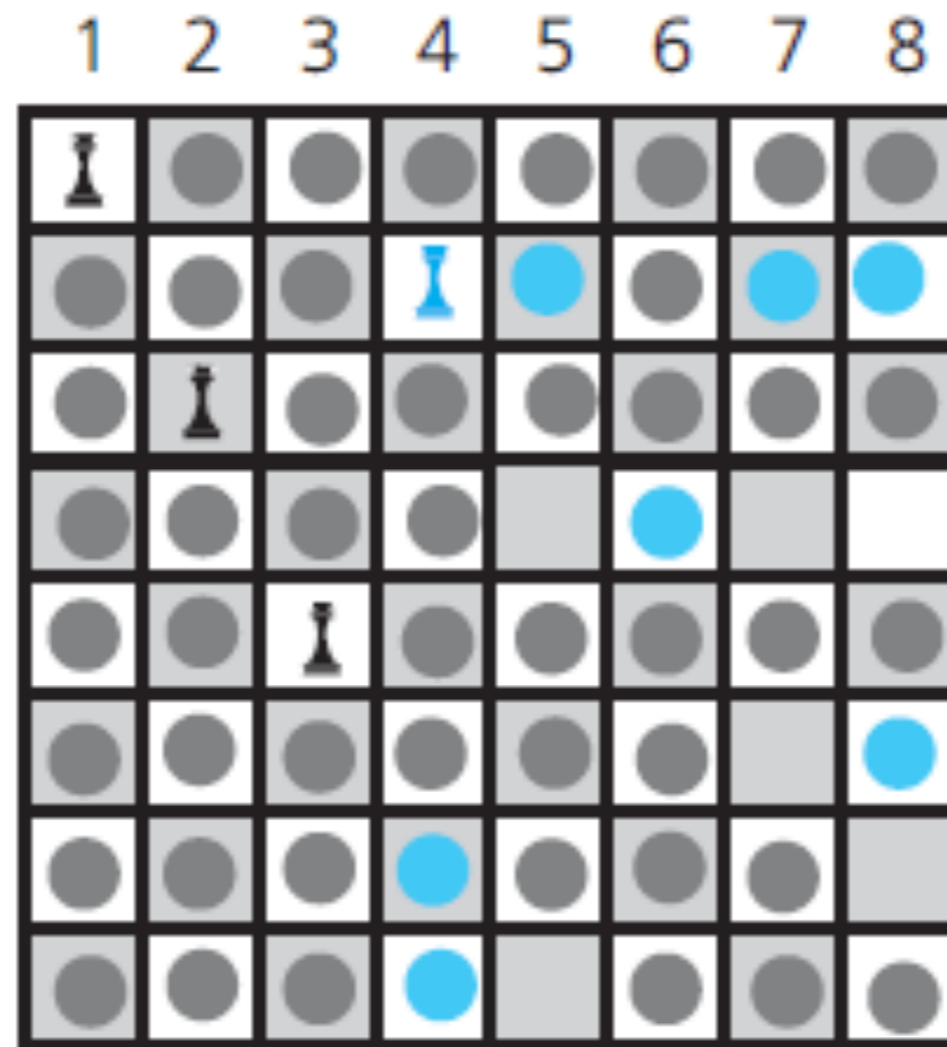


(b) The second queen in column 2

The Eight Queens Problem

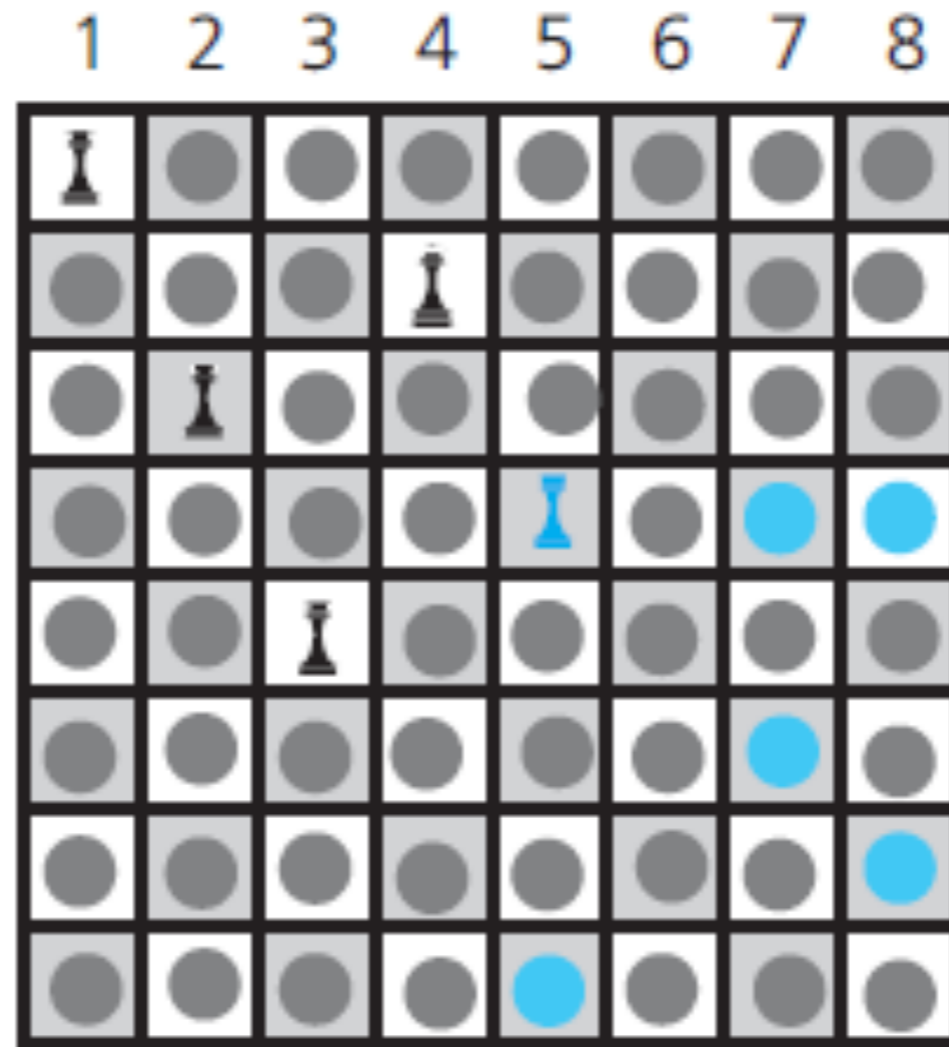


The Eight Queens Problem



(d) The fourth queen in column 4

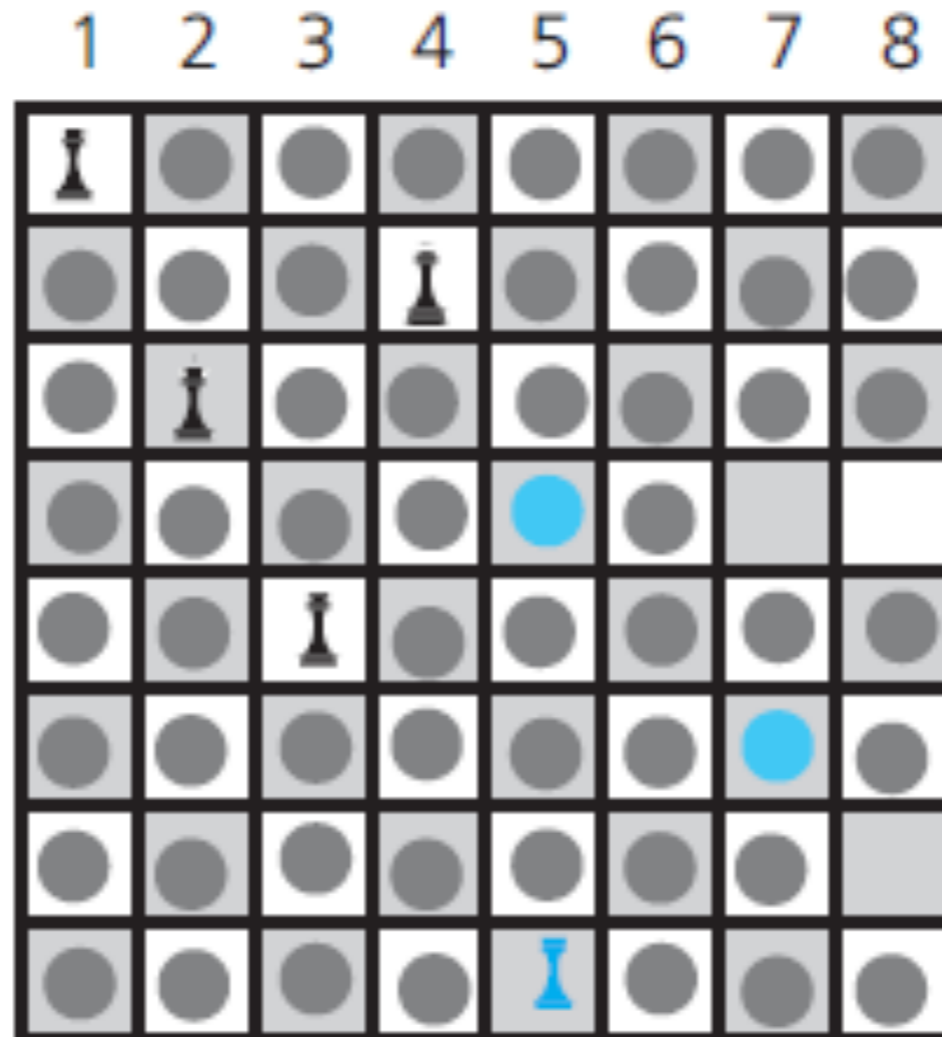
The Eight Queens Problem



(e) The five queens
can attack all of column 6

The Eight Queens Problem

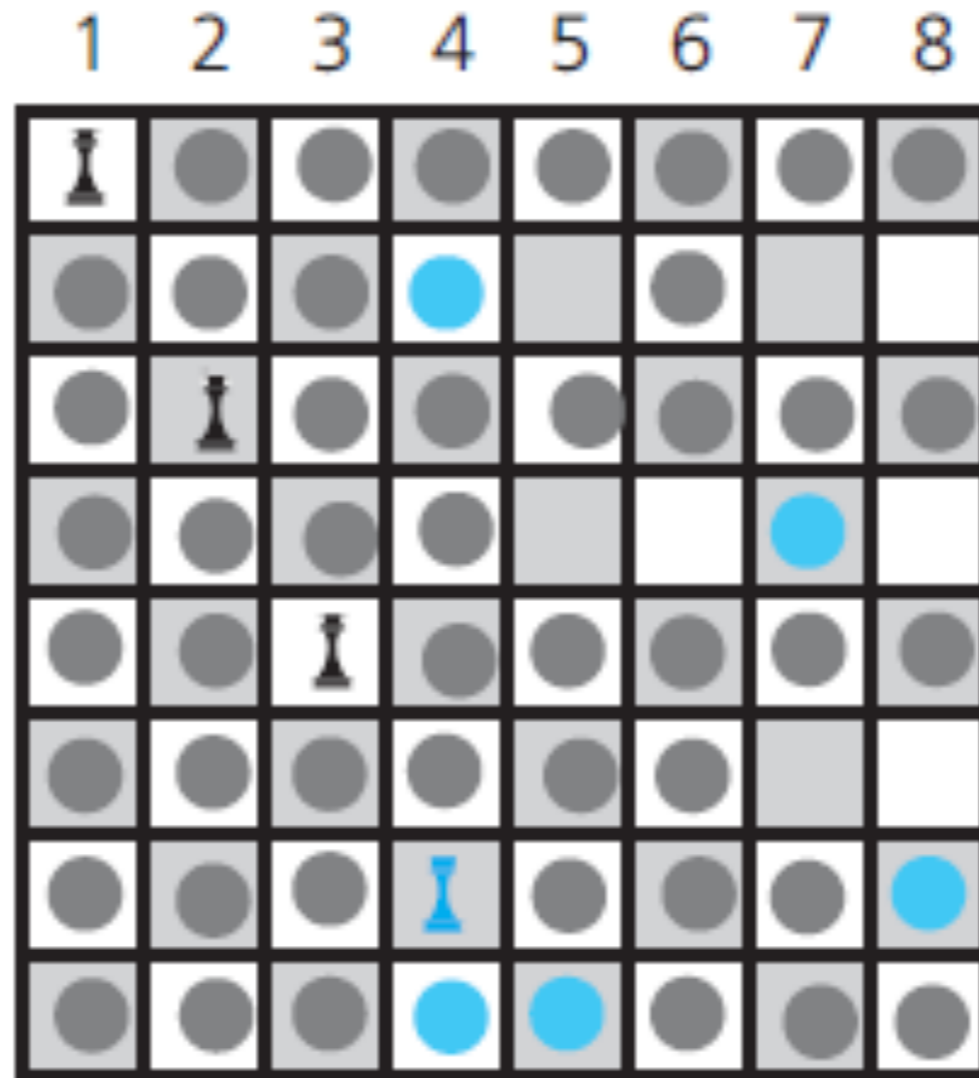
Backtracking!



(f) Backtracking to column 5 to try another square for the queen

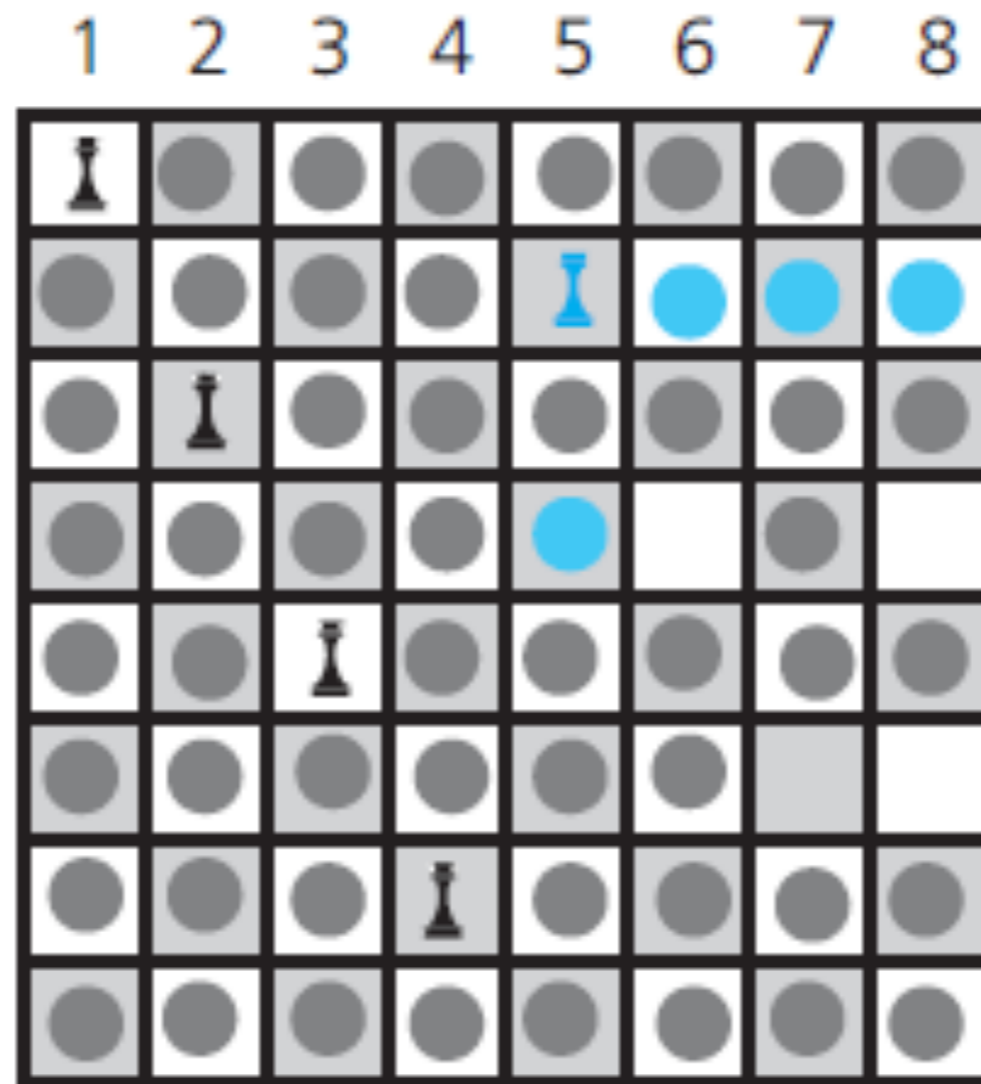
The Eight Queens Problem

Backtracking!



(g) Backtracking to column 4
to try another square for
the queen

The Eight Queens Problem



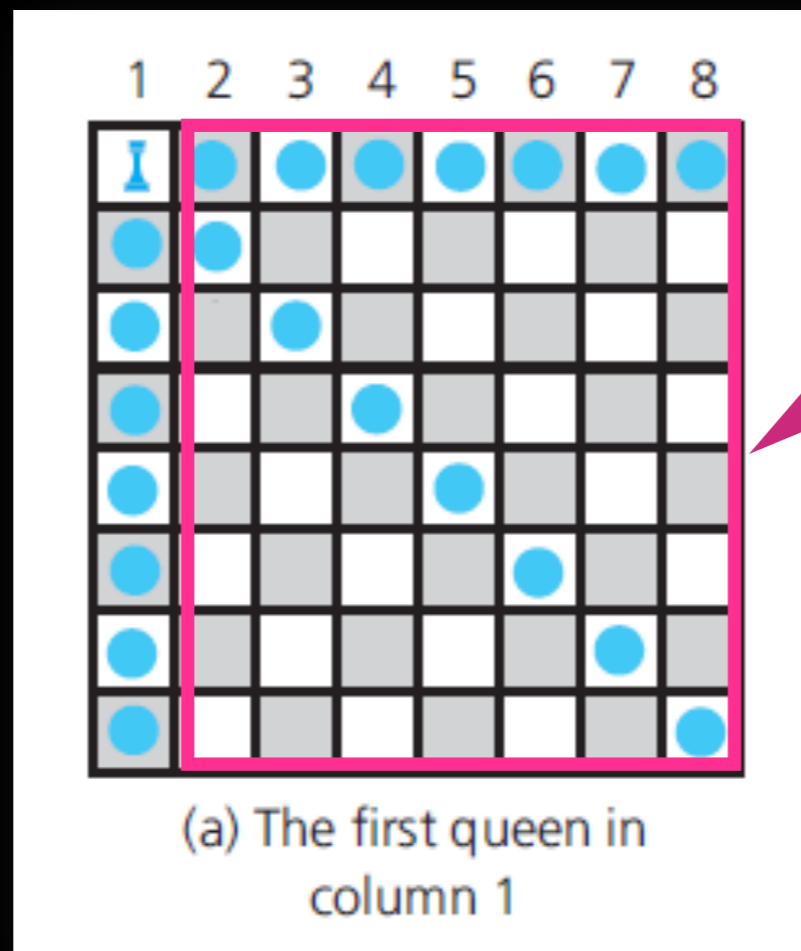
(h) Considering column
5 again

The Eight Queens Problem

How can we express this problem recursively?

The Eight Queens Problem

How can we express this problem recursively?



Place queen on column i
Recursively solve on
columns $(i+1)$ to 8

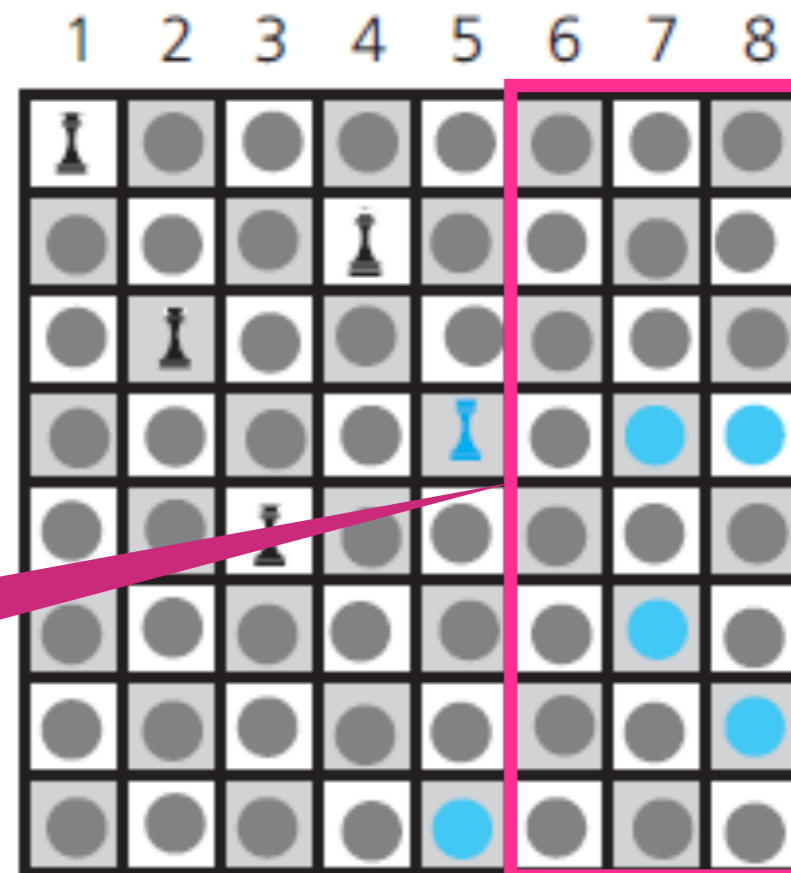
The Eight Queens Problem

How do we backtrack?

The Eight Queens Problem

How do we backtrack?

Communicate to calling function that there are no options left, it should try something else!



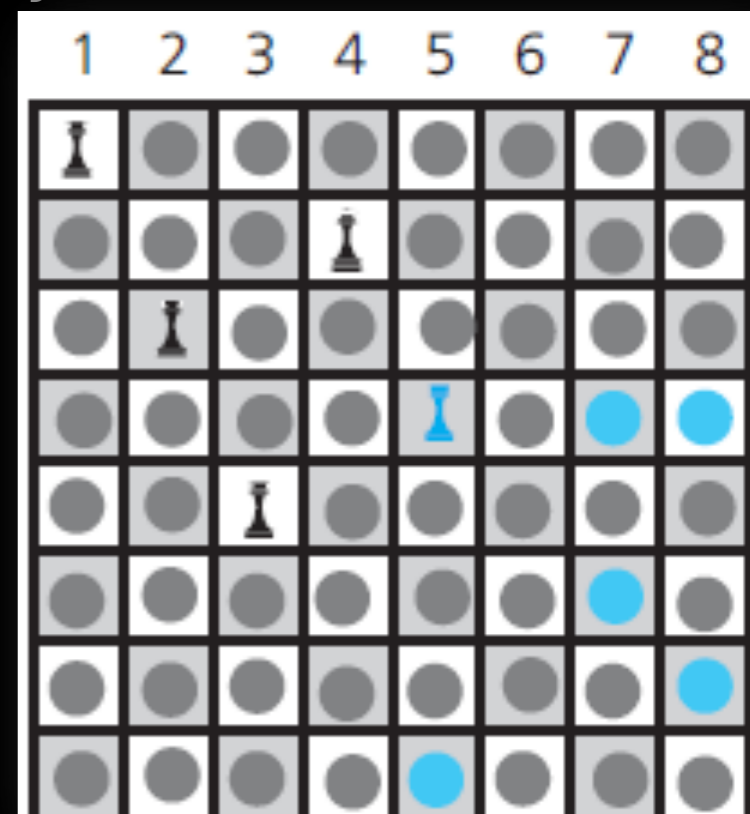
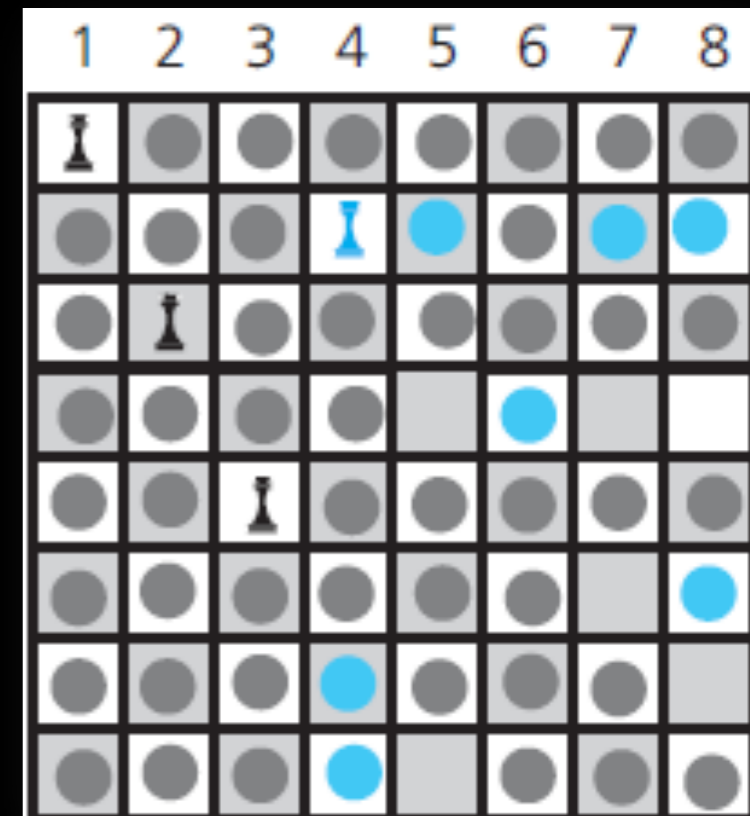
(e) The five queens can attack all of column 6

The Eight Queens Problem

```

bool placeQueens(board, column)
{
    if(column > BOARD_SIZE)
        return true; //Problem is solved!
    else
    {
        while(there are safe squares in this column)
        {
            //place queen in next safe square;
            if(placeQueen(board, column+1)) //recursively look forward
                → return true; //queen safely placed
        }
        return false; //recursive backtracking
    }
}

```

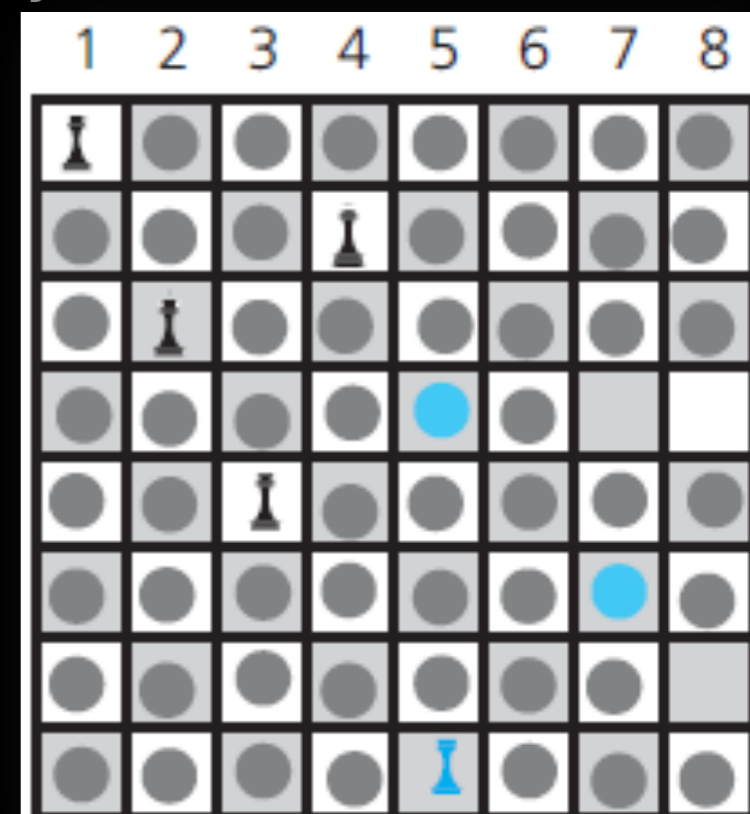
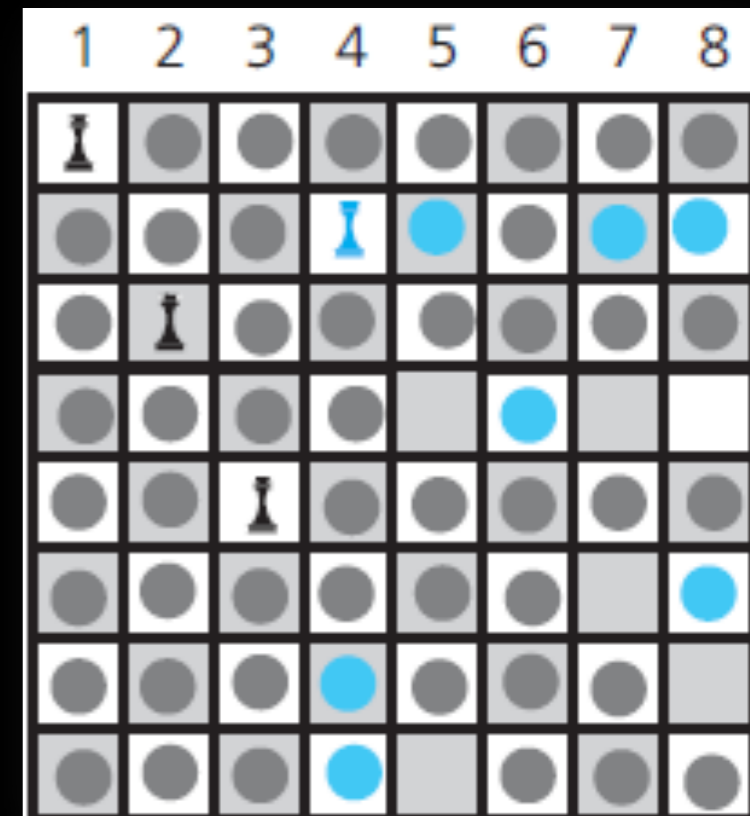


The Eight Queens Problem

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}

```



Think Algorithmically

*“Experienced Computer Scientists analyze and solve computational problems at a level of **abstraction** that is beyond that of any particular programming language / representation / implementation”*

Algorithm Design

- Identify the problem (input, output, states)
- Come up with a procedure that will lead to solution
- Independent of implementation detail



Initial phase/step

Model your problem/data

- **represent** the problem to support your algorithm

Implement solution

- Language
- Data structure
- Implementation detail

Think Algorithmically

- Takes practice
- The more you see/do the easier it gets
- There are some frameworks that can guide you, we have seen only a few, you will continue to learn more throughout your career

E.g.

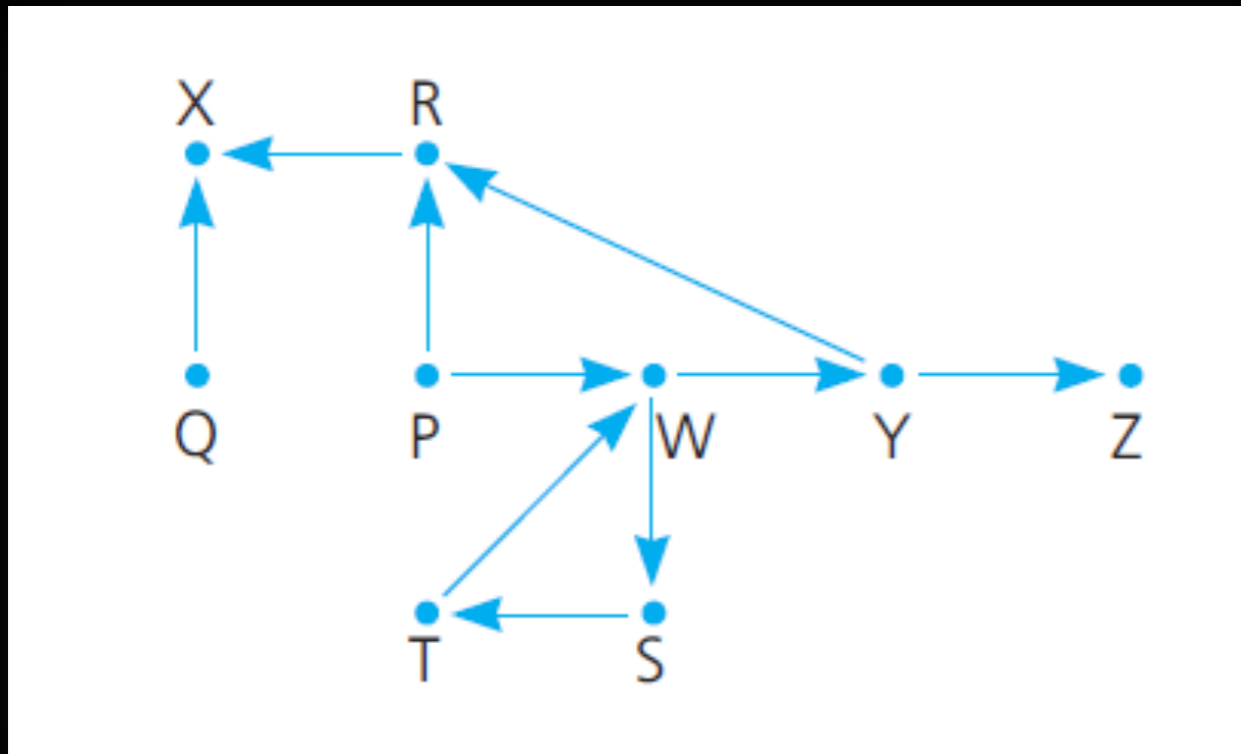
- Can I cast this as a backtracking problem?
- Can I cast this as a decision-making / decision tree problem?
- Do I need to enumerate all solutions?
- Am I looking for best/optimal solution?

Lecture Activity

Write **PSEUDOCODE** for a **RECURSIVE** function that finds a path from origin to destination.
Assume cities are visited in alphabetical order.

```
bool findPath(map, origin, destination)
```

Origin = P , **Destination = Z**



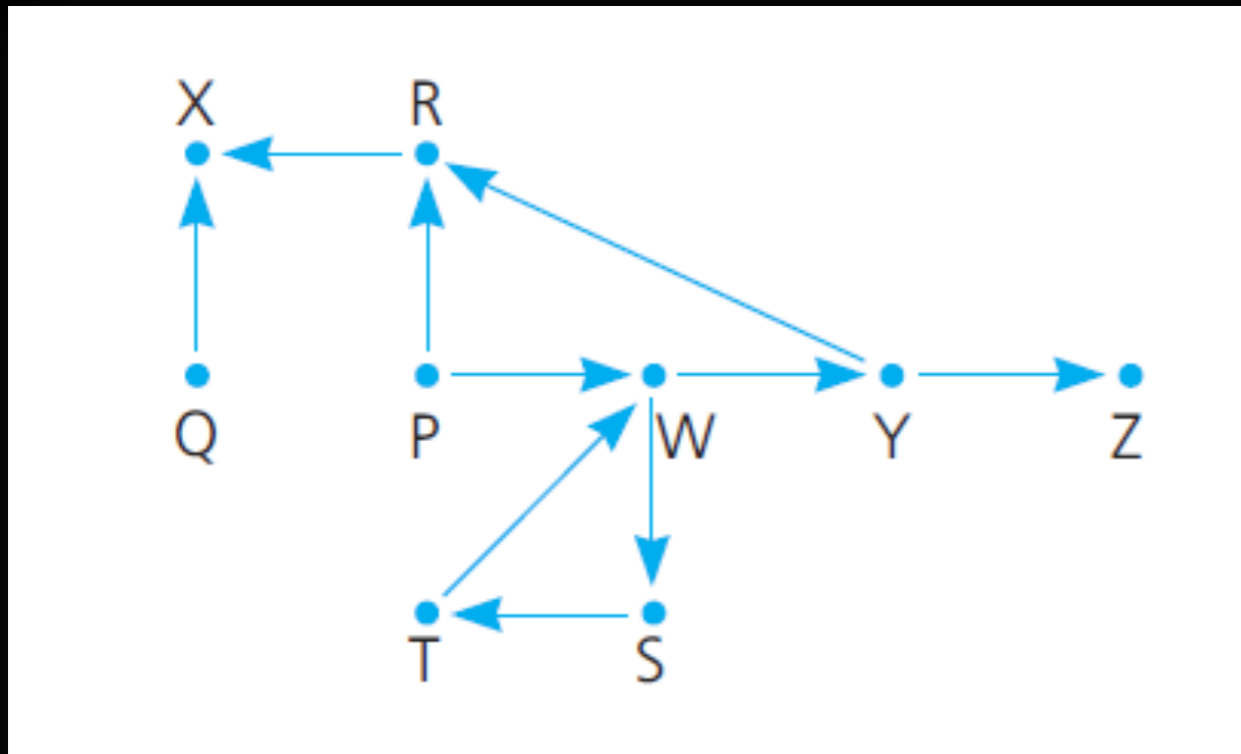
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P

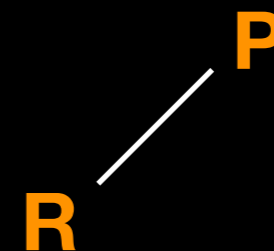
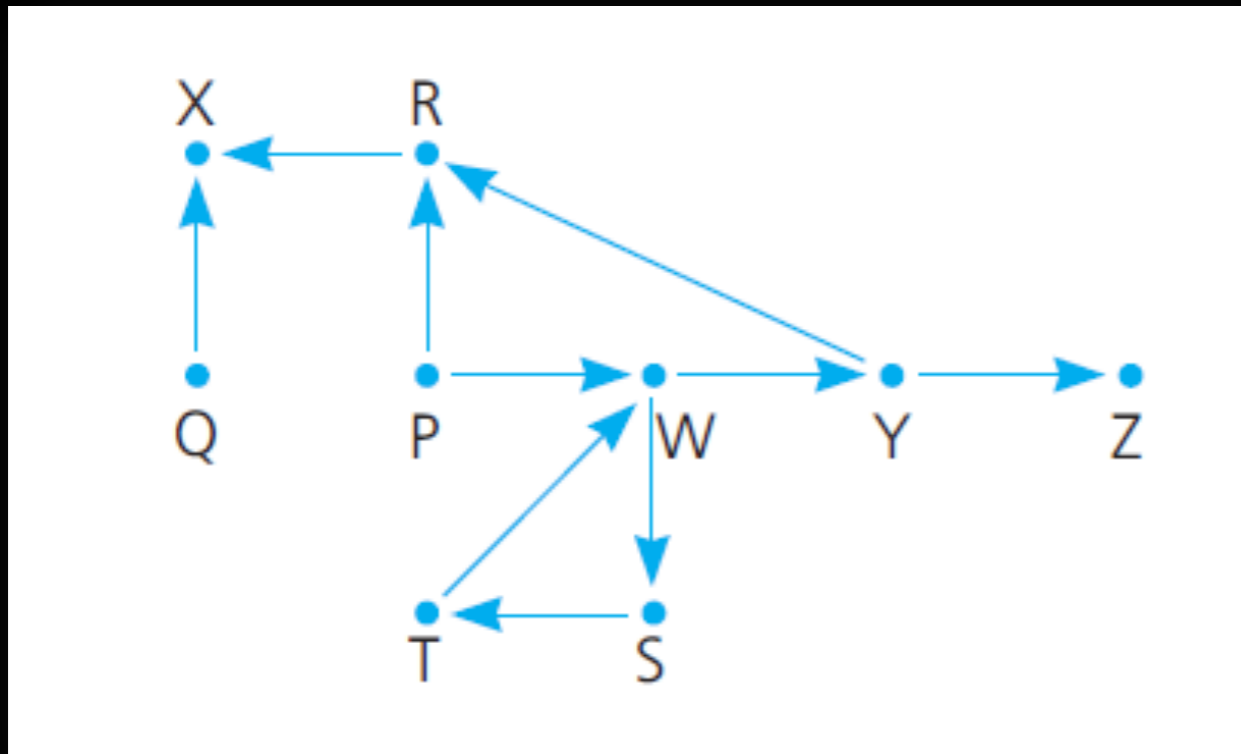


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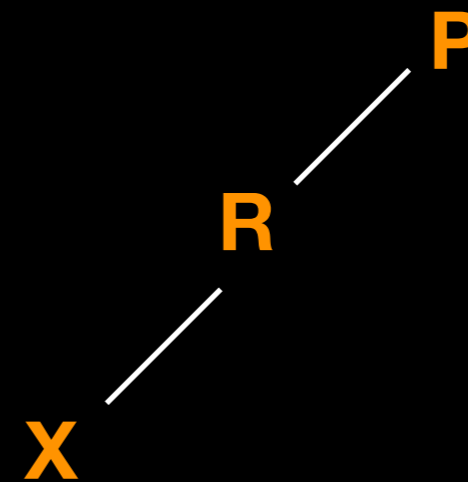
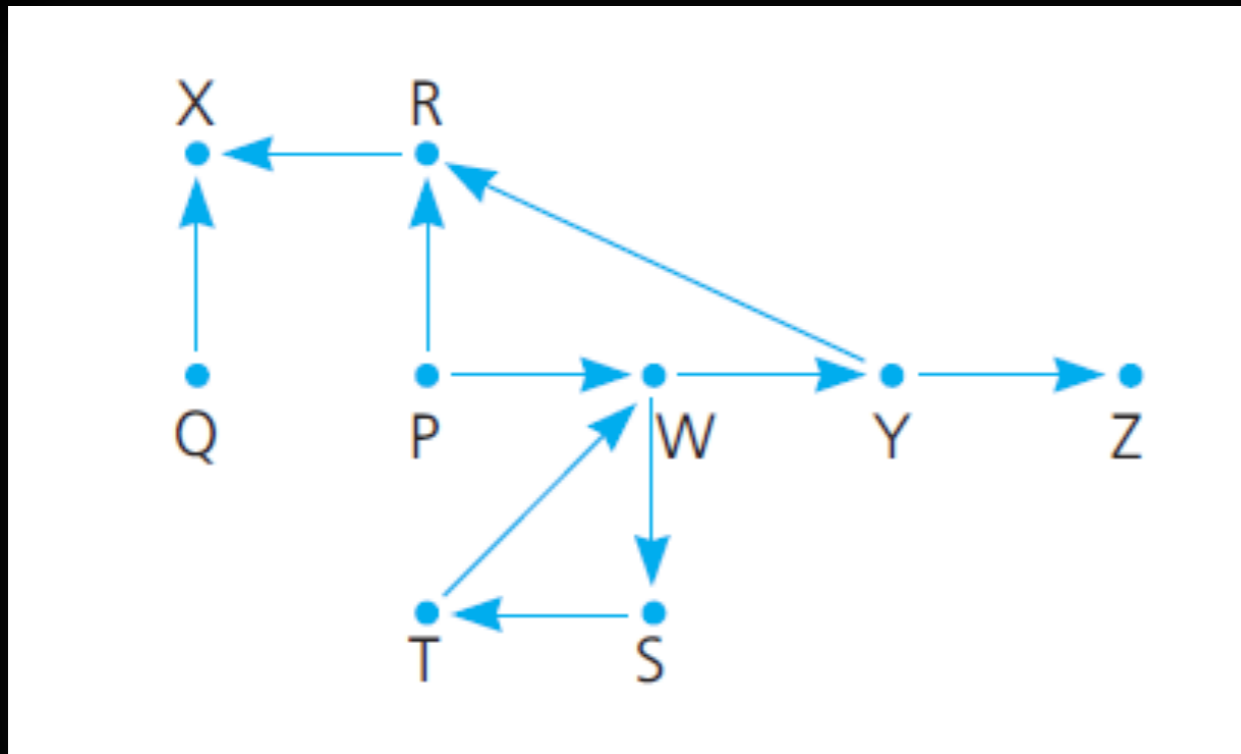


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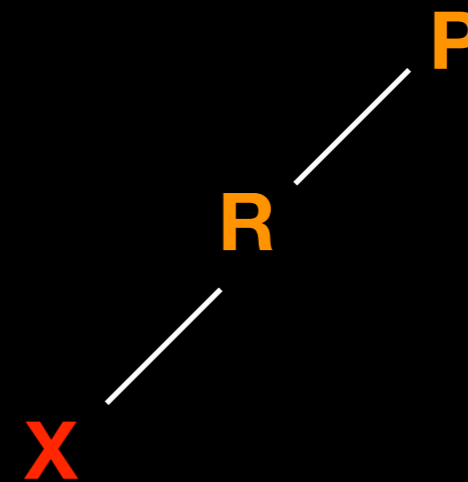
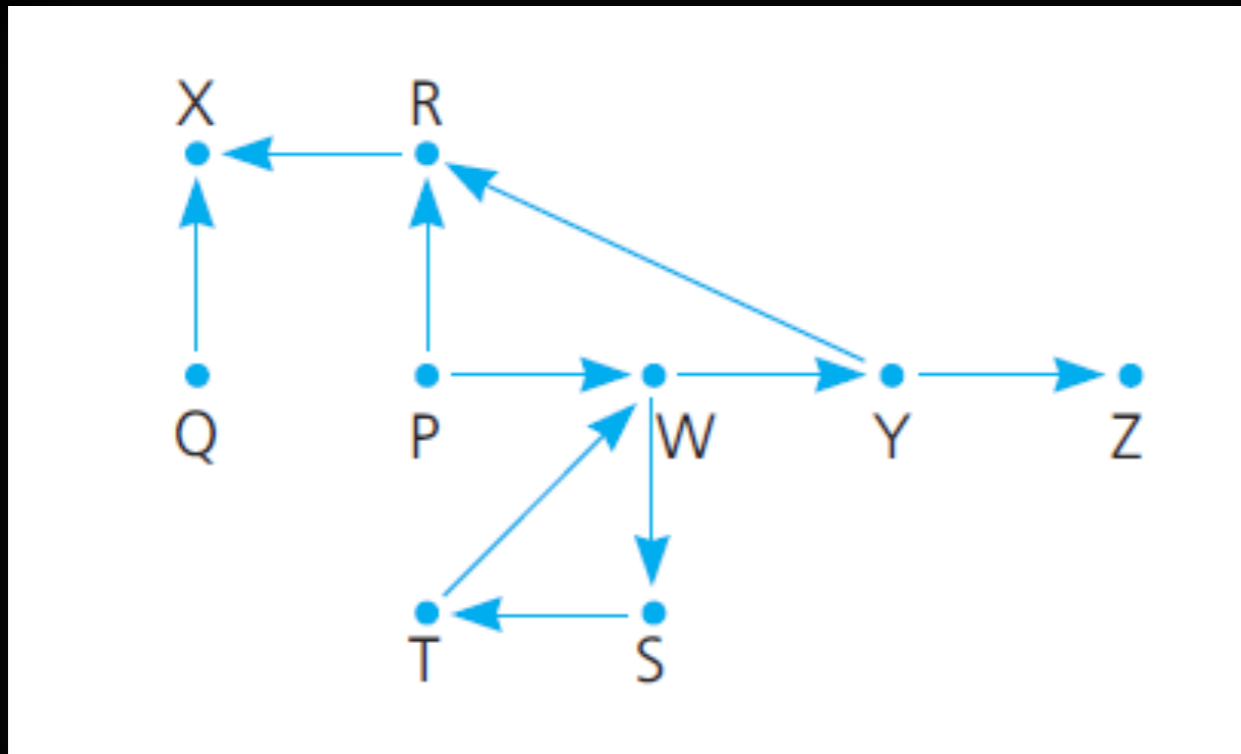


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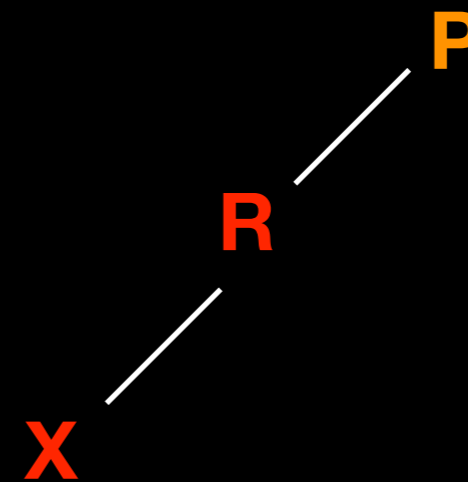
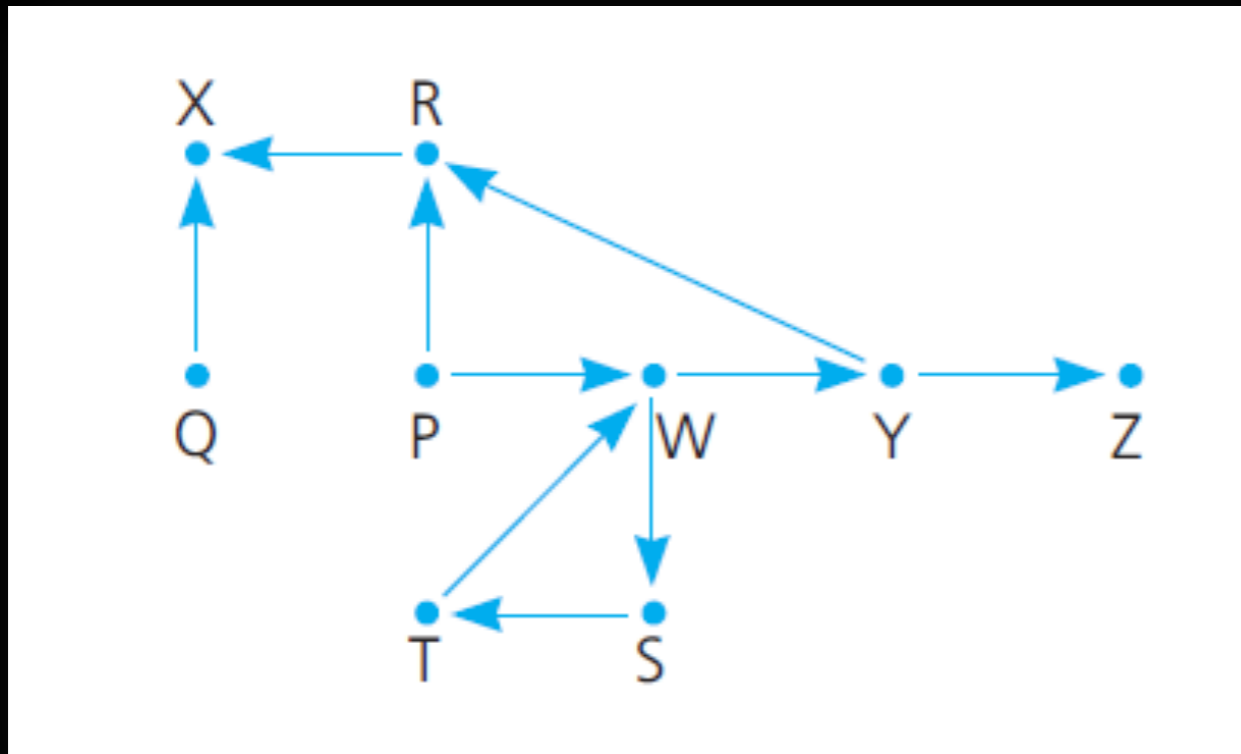


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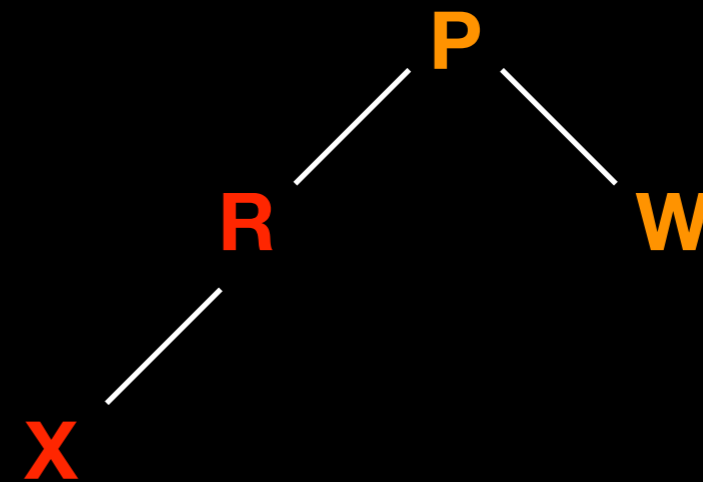
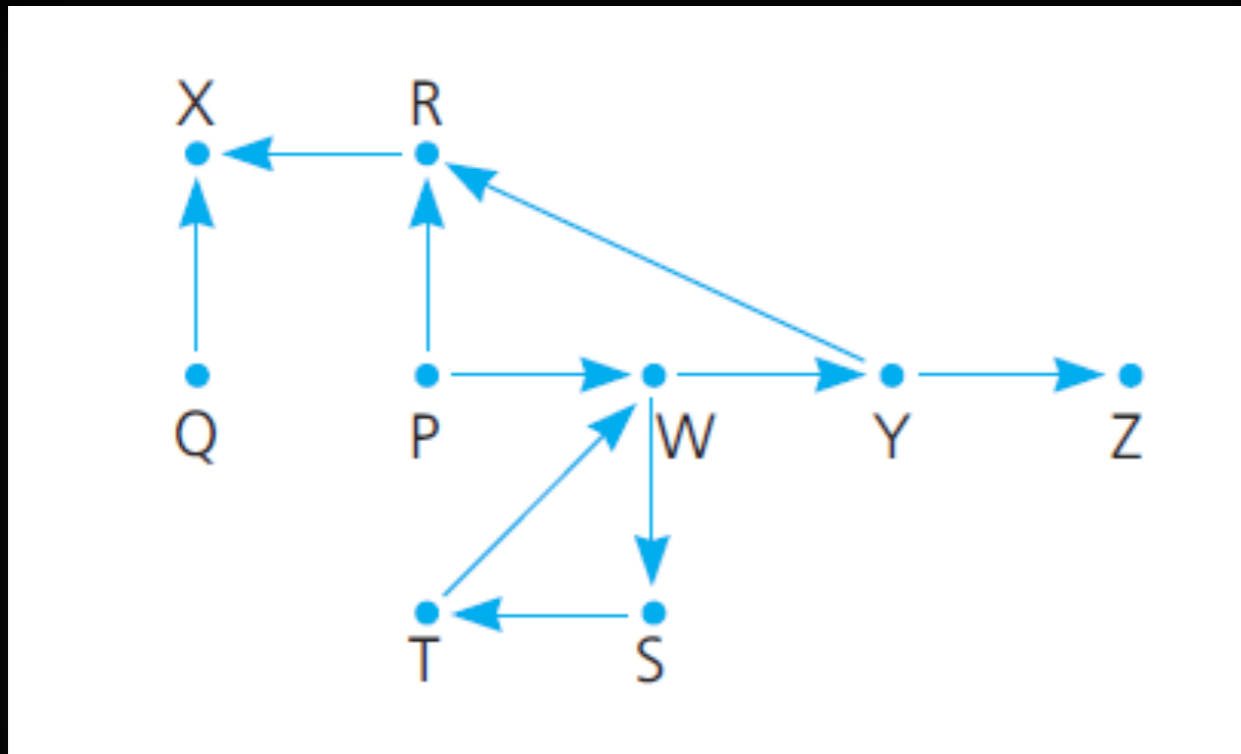


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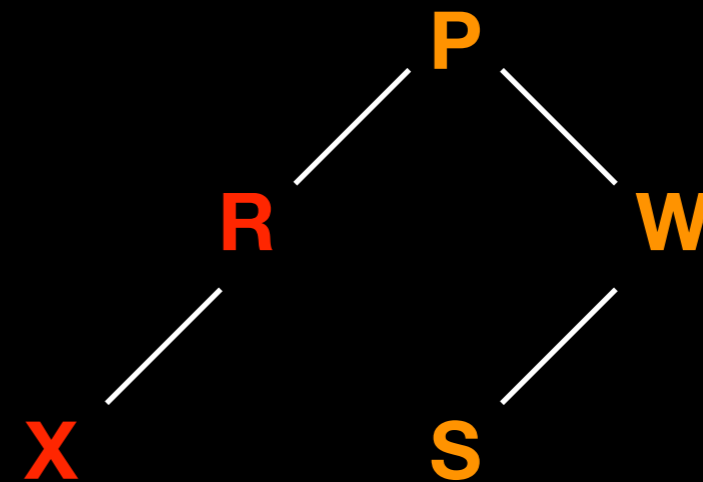
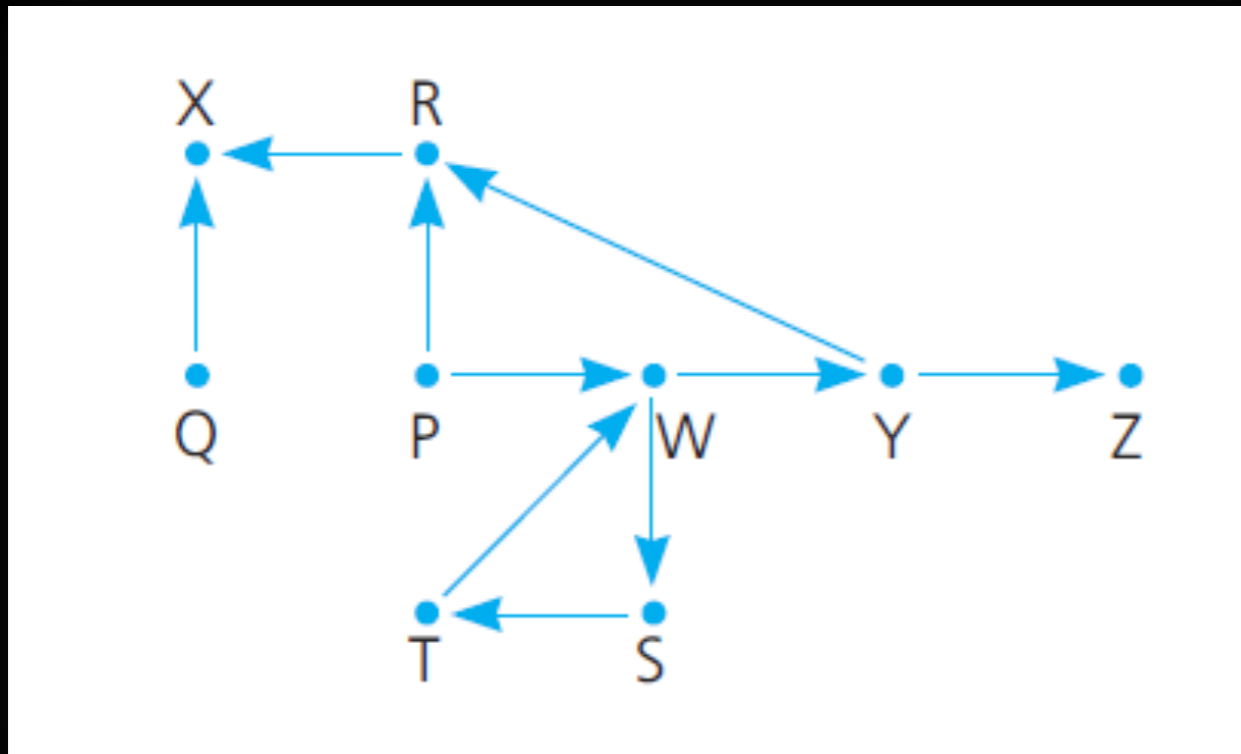


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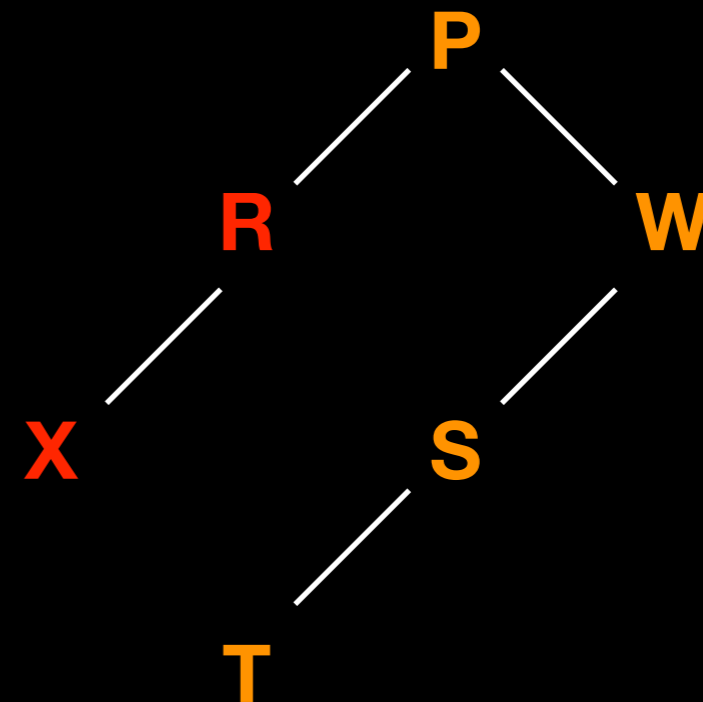
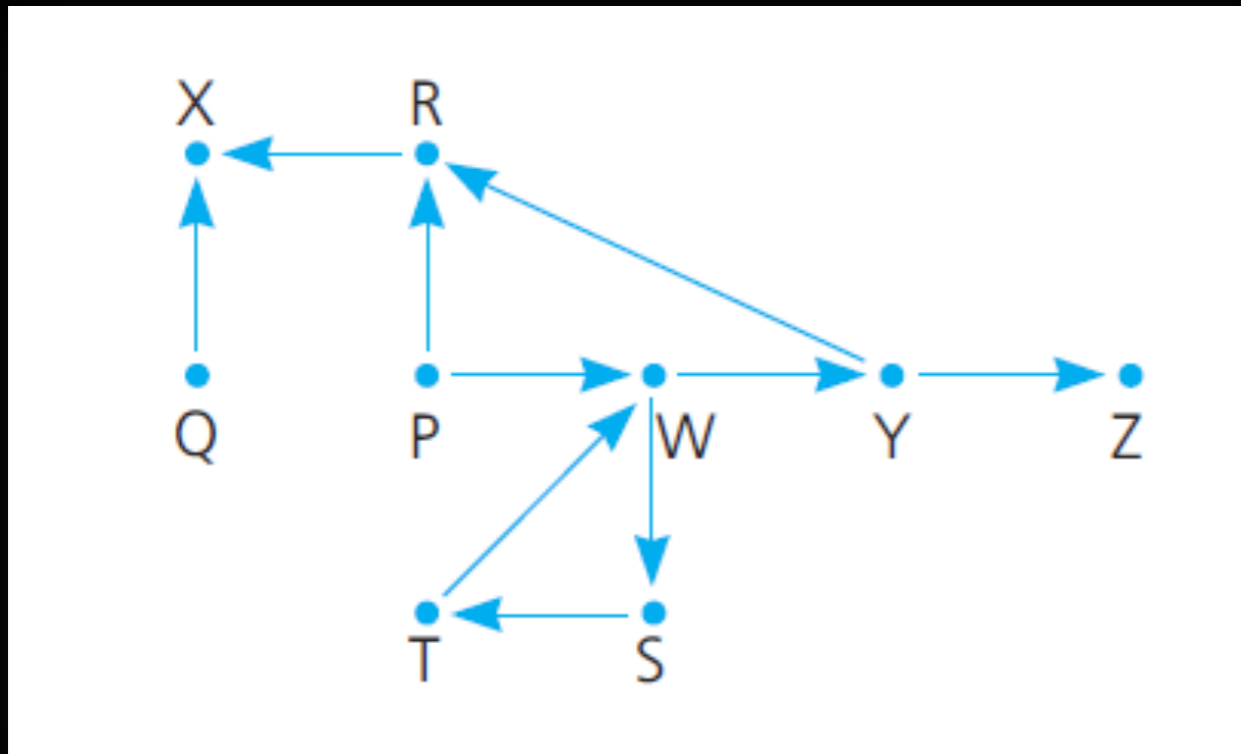


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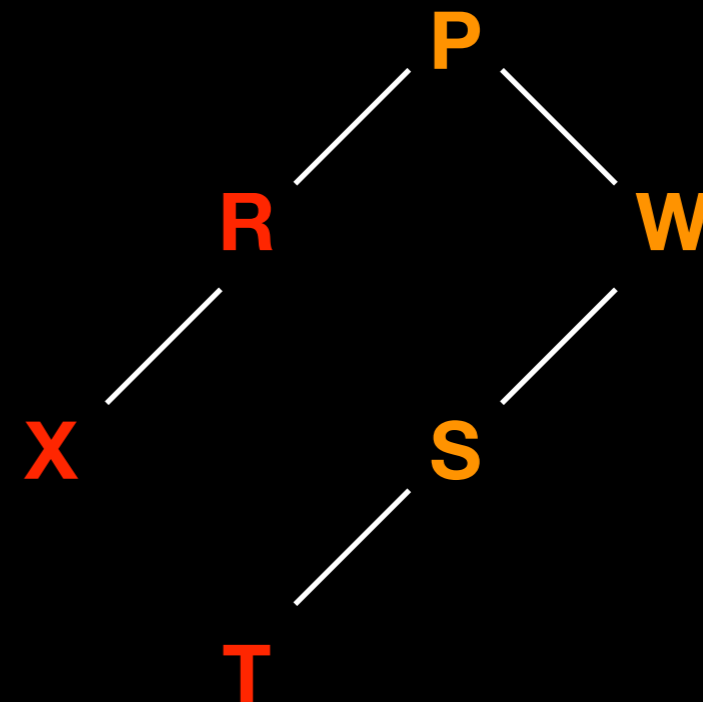
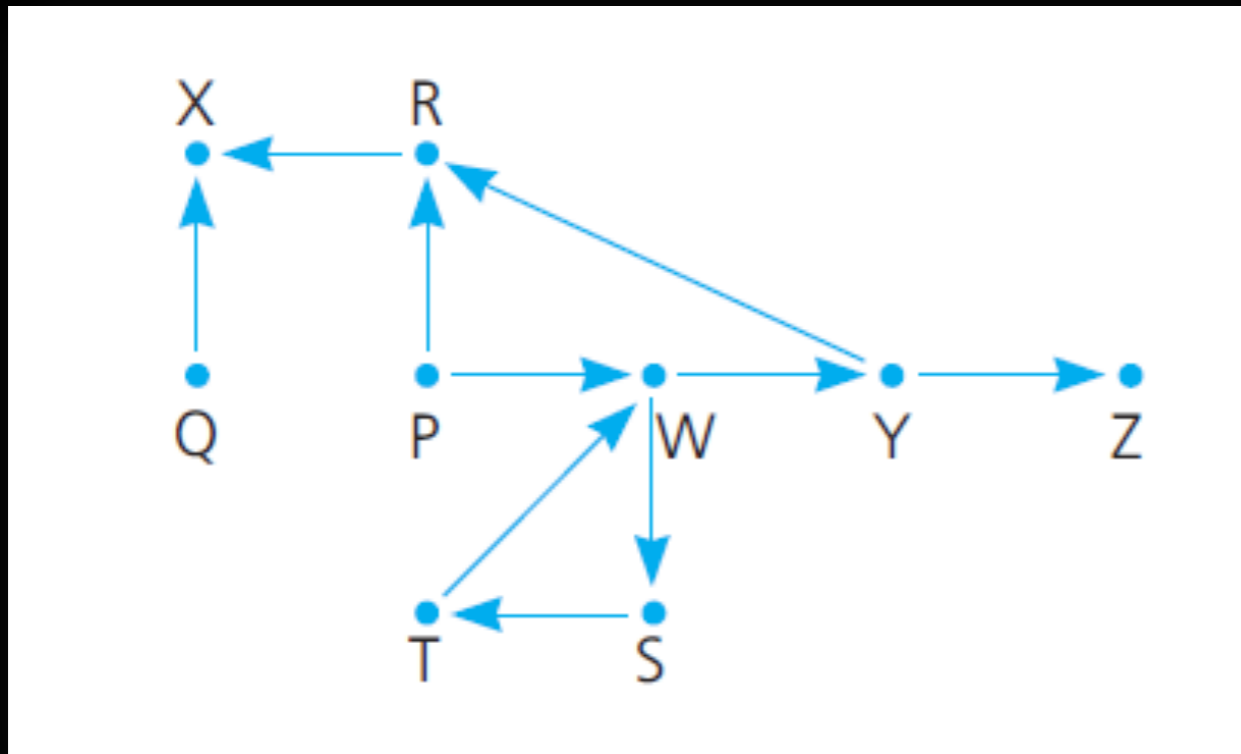


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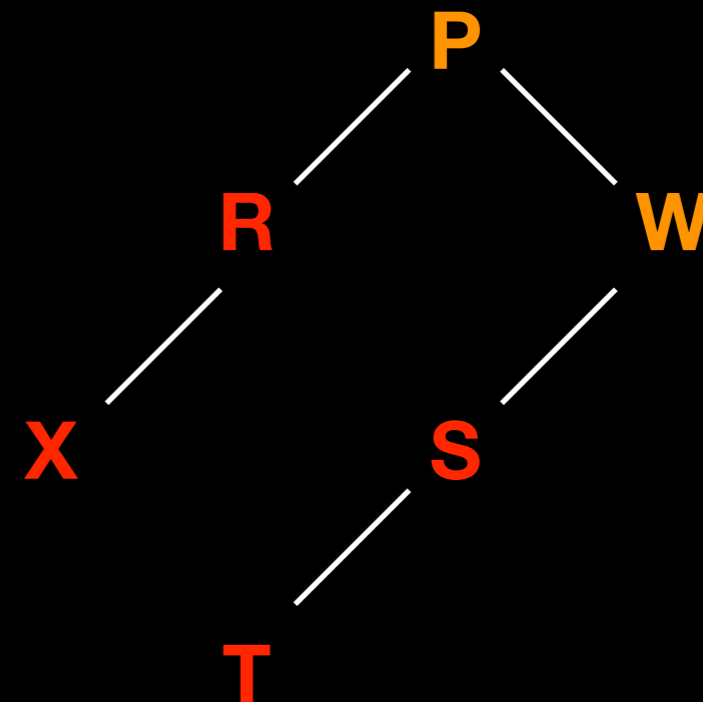
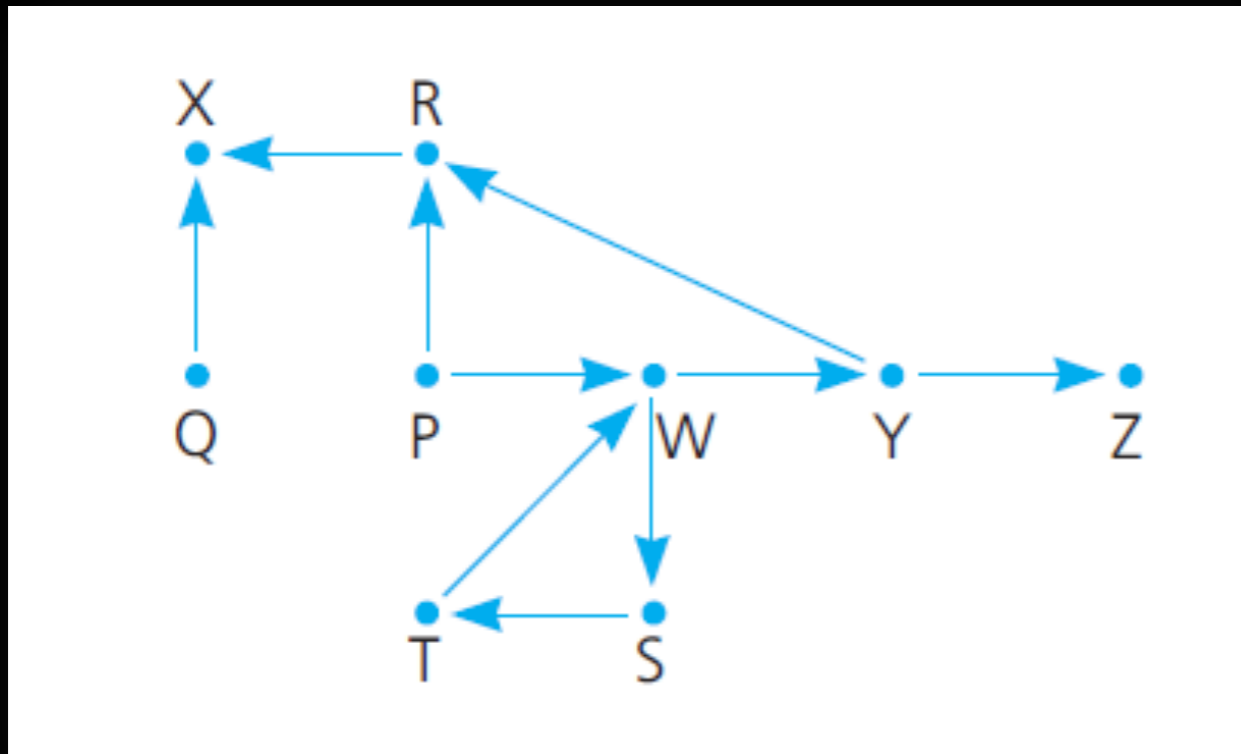


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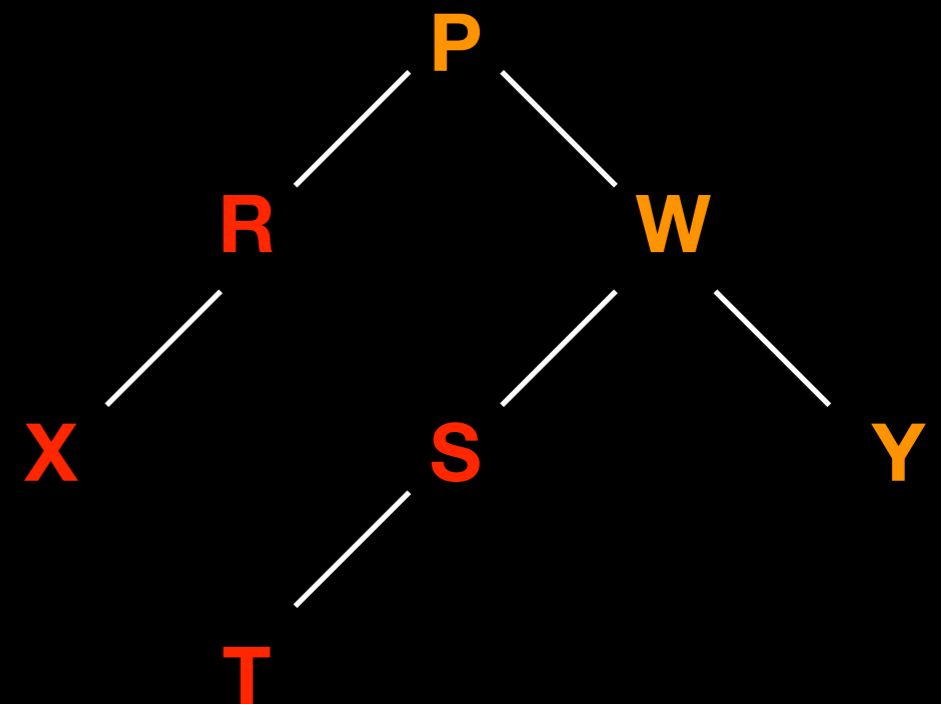
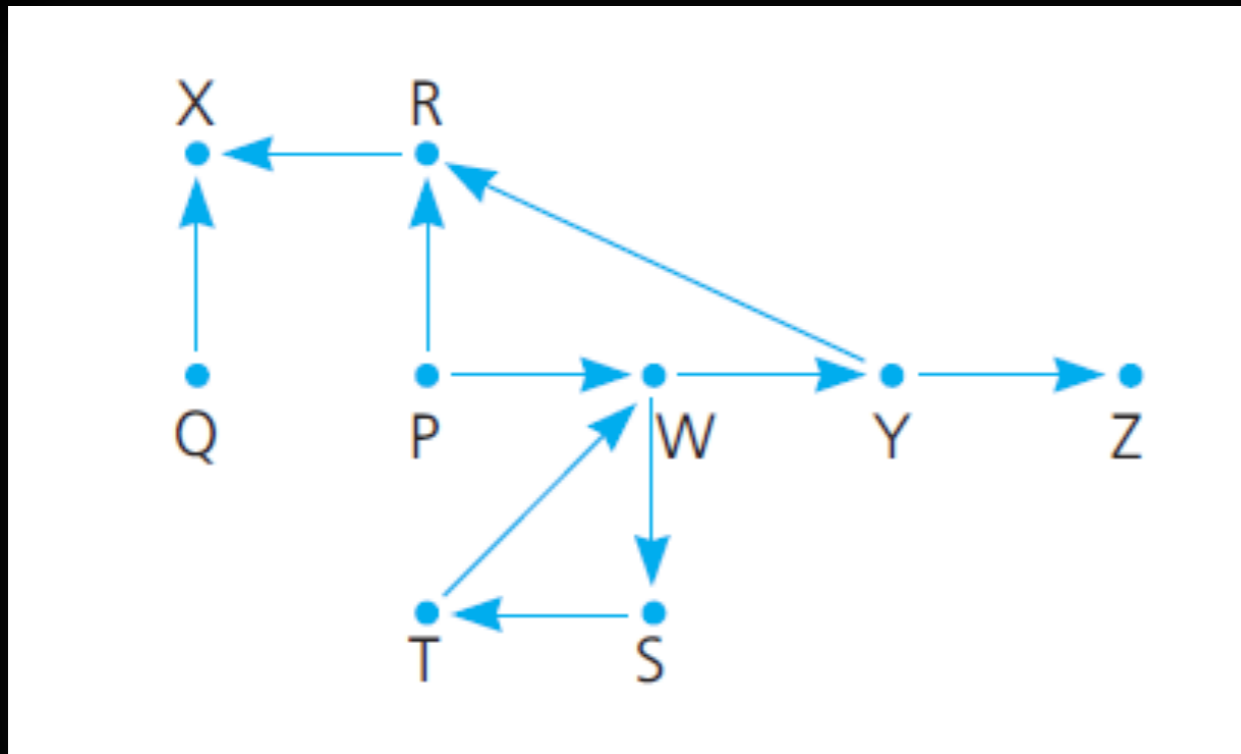


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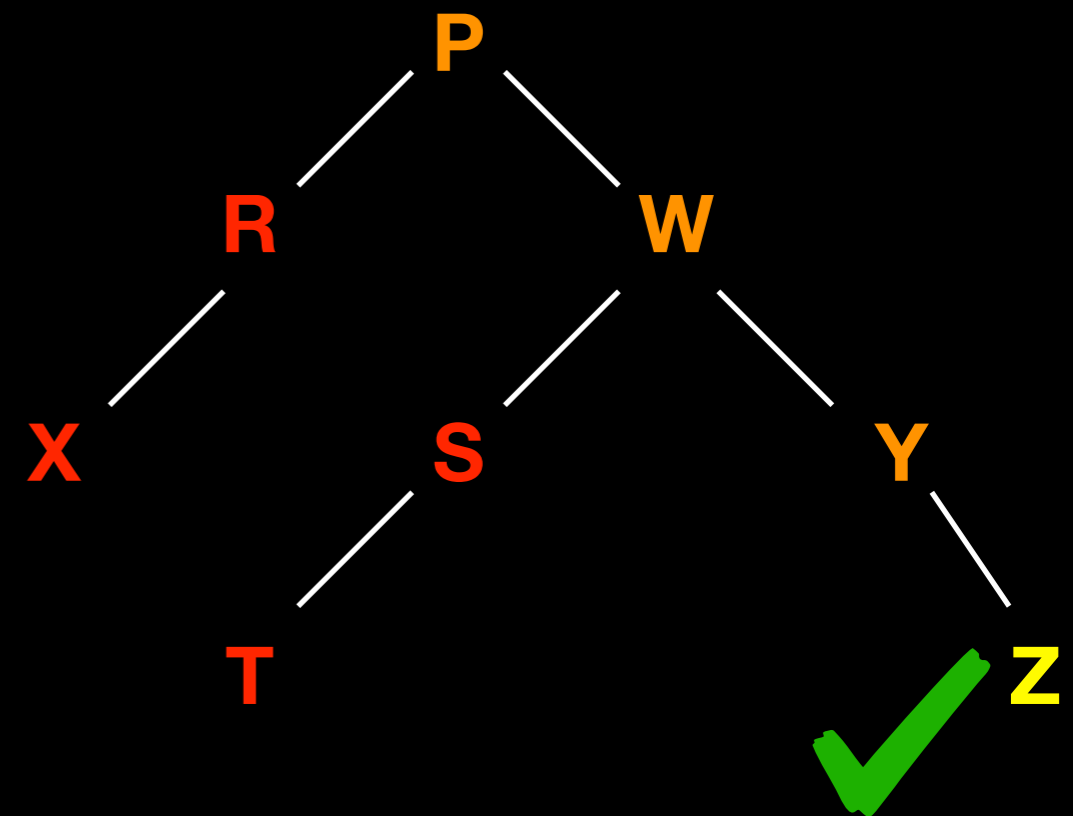
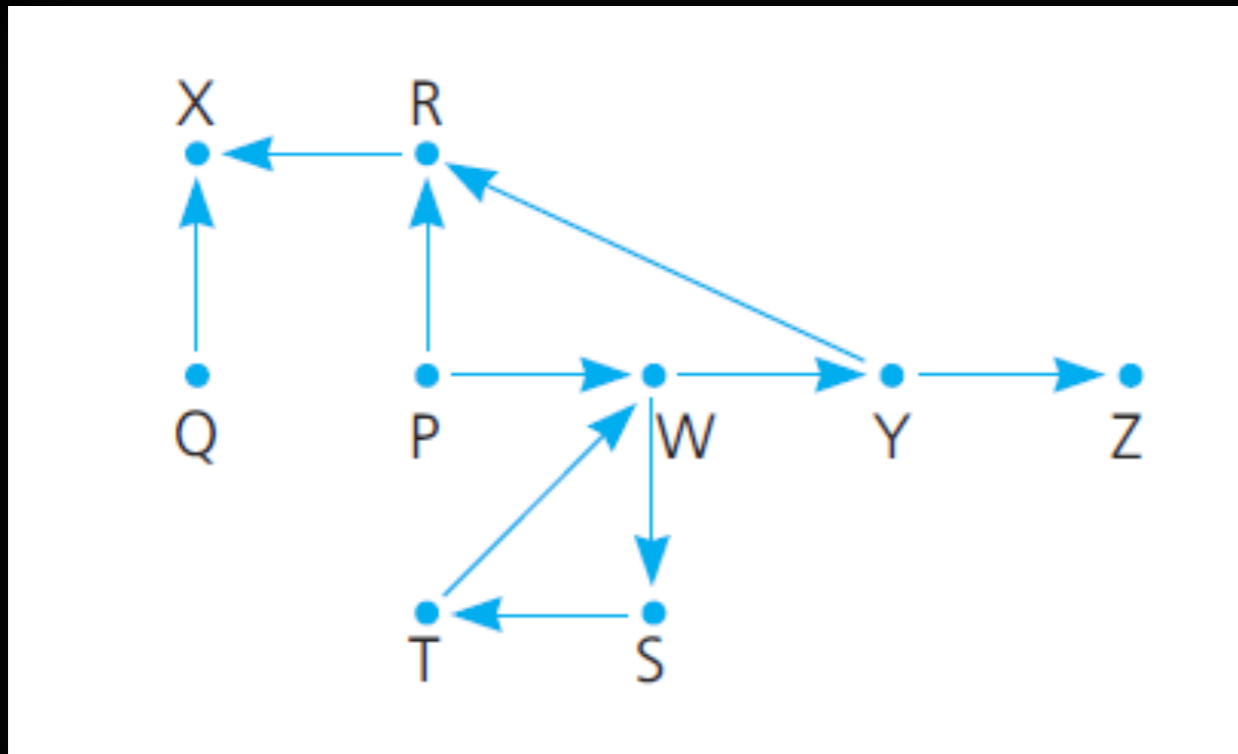


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Lecture Activity

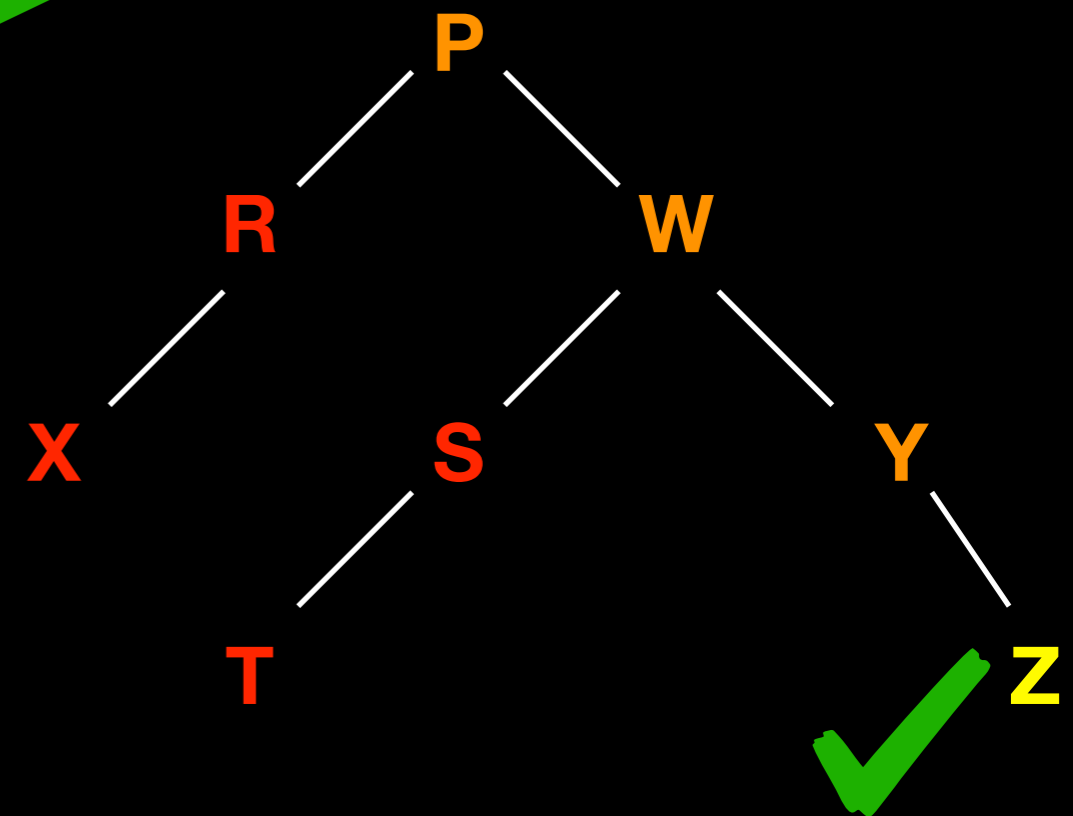
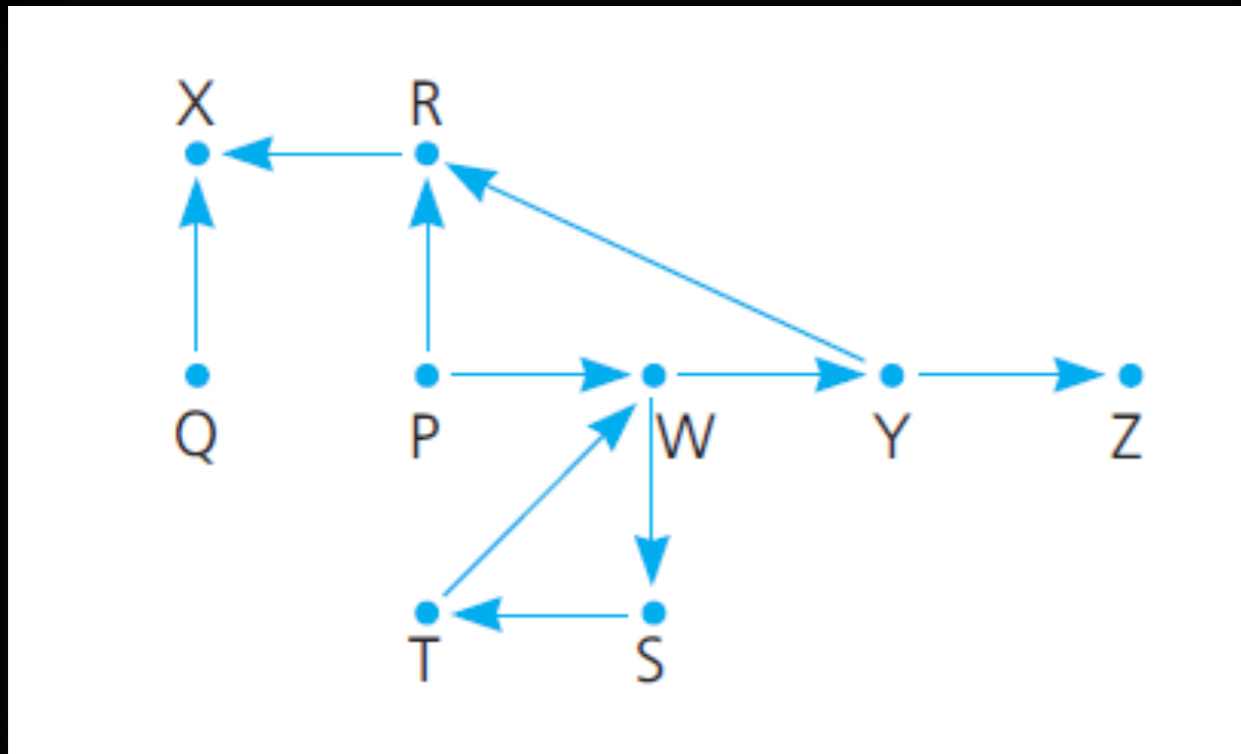
Write **PSEUDOCODE** for a **RECURSIVE** function that finds a path
Assume cities are visited in alphabetical order.

```
bool findPath(map, origin, destination)
```

Don't get bogged down
by what a map is.

In design phase you
know it's available and
you can look up where
you can go next from

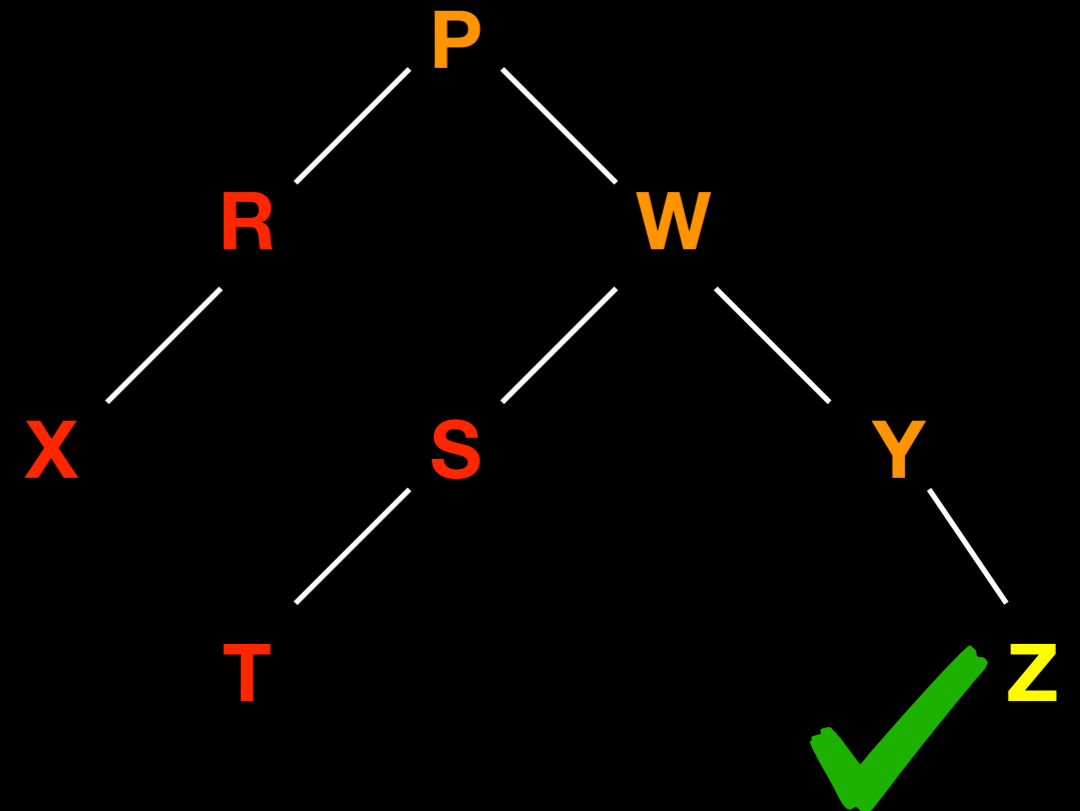
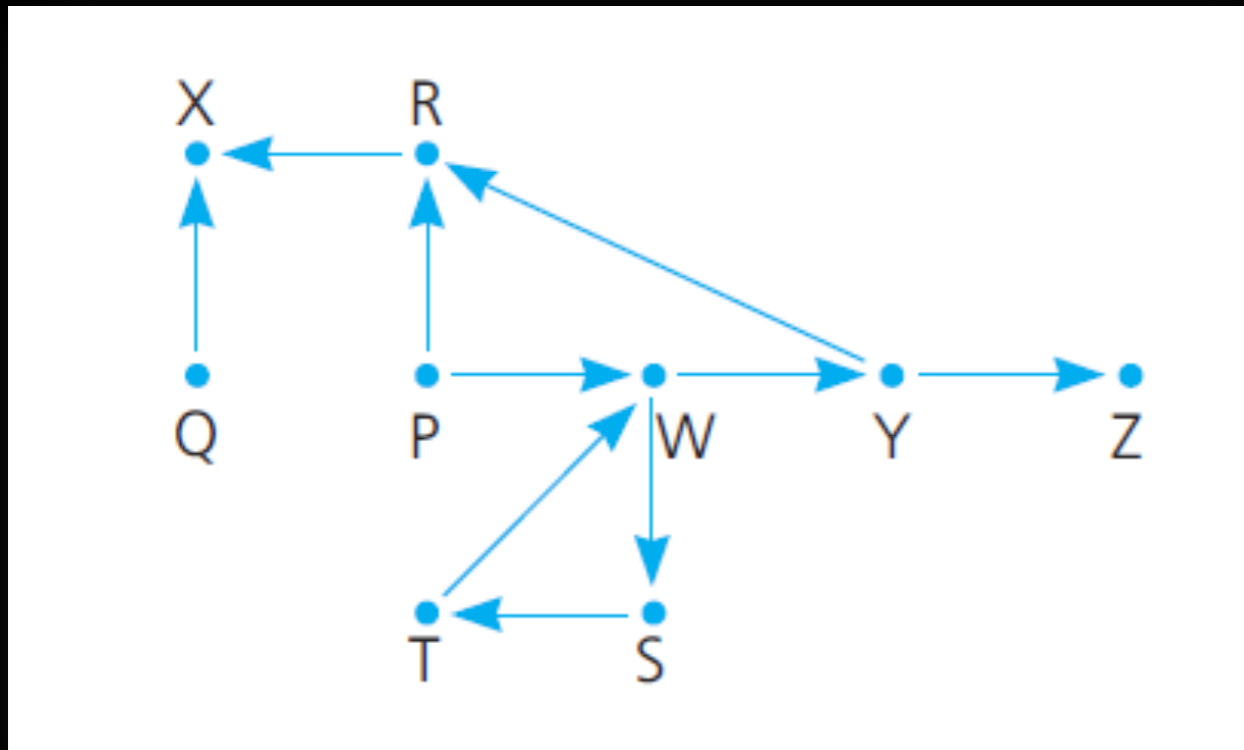
Origin = P , **Destination = Z**



Lecture Activity

```
bool findPath(map, origin, destination)
{
    mark origin as visited in map
    if origin == destination
        return true
    else
        for each unvisited city C reachable from origin
            if findPath(map, C, destination) ← Recursive call
                return true
        return false //recursive backtracking
}
```

Origin = P , **Destination = Z**



Find Permutations

Toy example to
make initial
observation

A	B	C	D
---	---	---	---

Order Matters!

A	B	C	D	B	A	C	D	C	A	B	D	D	A	B	C
A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Find Permutations

A	B	C	D
---	---	---	---

A	B	C	D
A	B	D	C
A	C	B	D
A	C	D	B
A	D	B	C
A	D	C	B

B	A	C	D
B	A	D	C
B	C	A	D
B	C	D	A
B	D	A	C
B	D	C	A

C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
C	D	B	A

D	A	B	C
D	A	C	B
D	B	A	C
D	B	C	A
D	C	A	B
D	C	B	A

Find Permutations

A	B	C	D
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A	C	B	D
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A	D	B	C
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B	A	C	D
B	A	D	C
B	C	A	D
B	C	D	A
B	D	A	C
B	D	C	A

C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
C	D	B	A

D	A	B	C
D	A	C	B
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A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
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A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
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A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
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A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

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A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Find Permutations

A	B	C	D
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B	A	D	C
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B	D	A	C
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C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
C	D	B	A

D	A	B	C
D	A	C	B
D	B	A	C
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A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Lock the first letter
For each letter you
lock
Permute the rest

Find Permutations

A B C D

A	B	C	D	B	A	C	D	C	A	B	D	D	A	B	C
A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Lock the first letter
For each letter you
lock

Permute the rest

DECISION

RECURSION

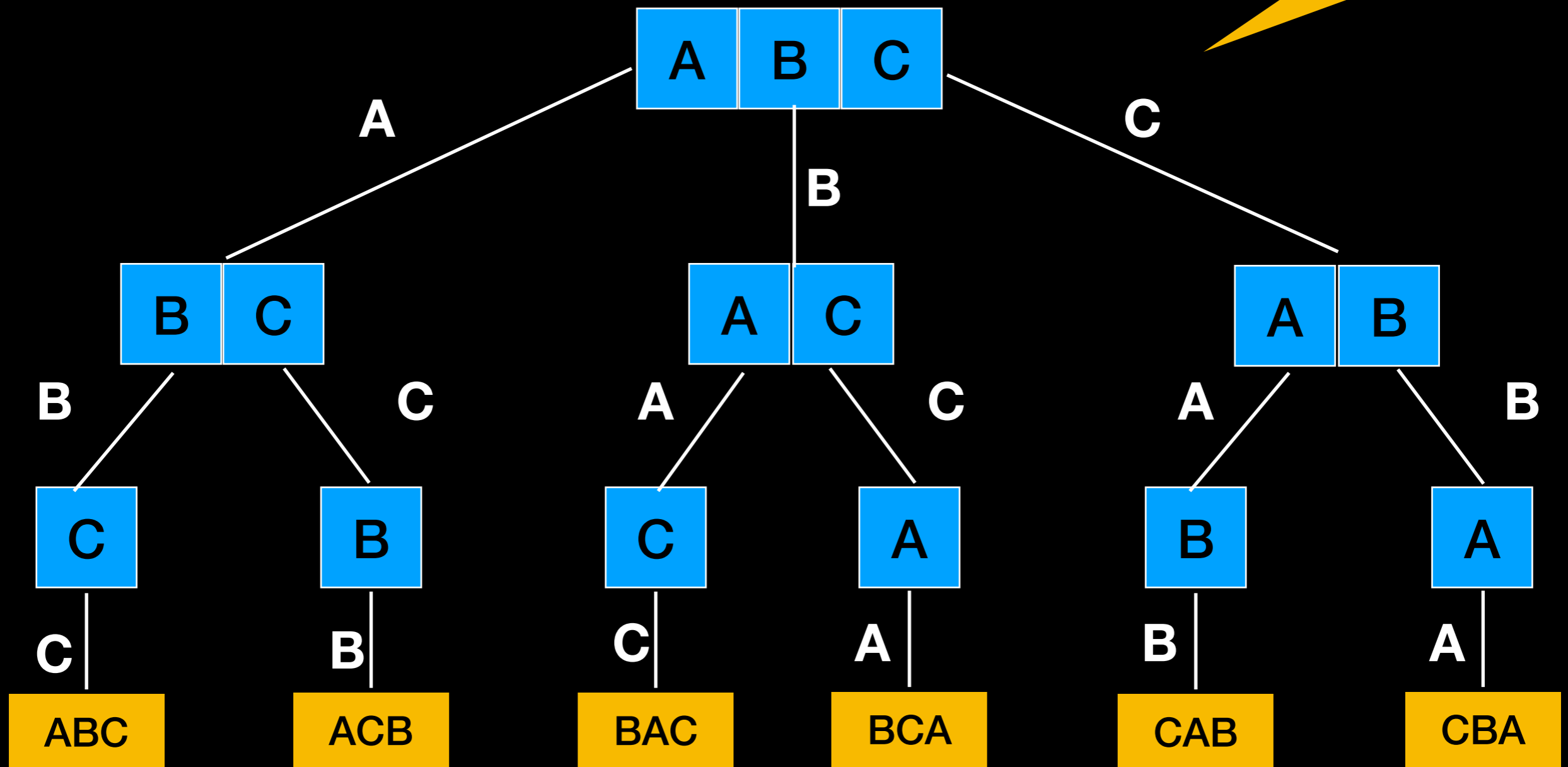
Find Permutations

A	B	C	D
---	---	---	---

A	B	C	D	B	A	C	D	C	A	B	D	D	A	B	C
A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Find Permutations

A Decision Tree

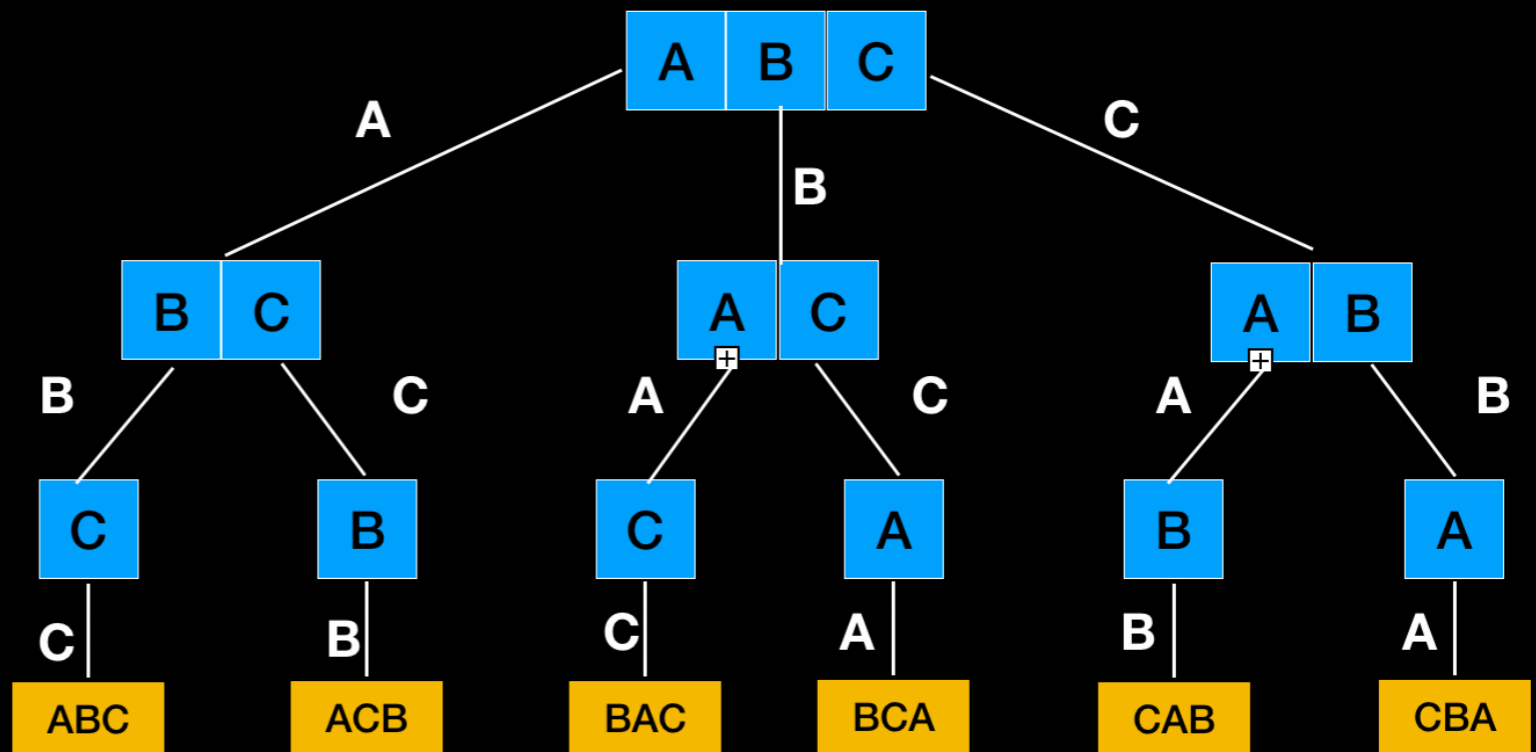



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    {
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        {
            swap(str[first],str[i]); //swap other characters with current first
            → permuteStr(str, first+1, last);
            swap(str[first],str[i]); //restore first char
        }
    }
}

```

ABC
BAC
CBA



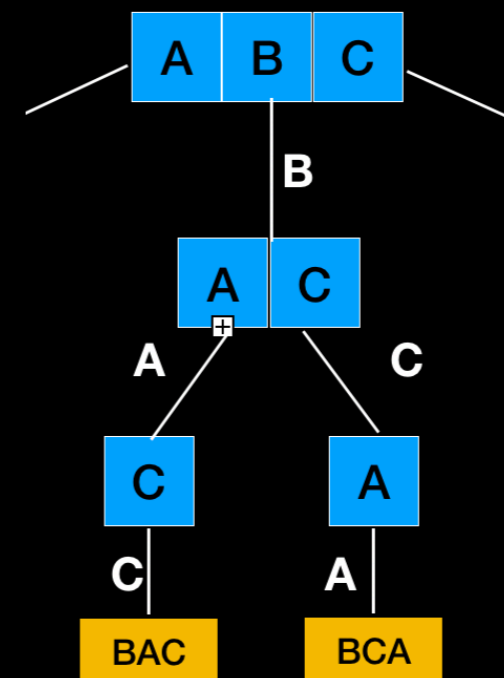
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        }
    }
}

```

i=1, first=0

ABC



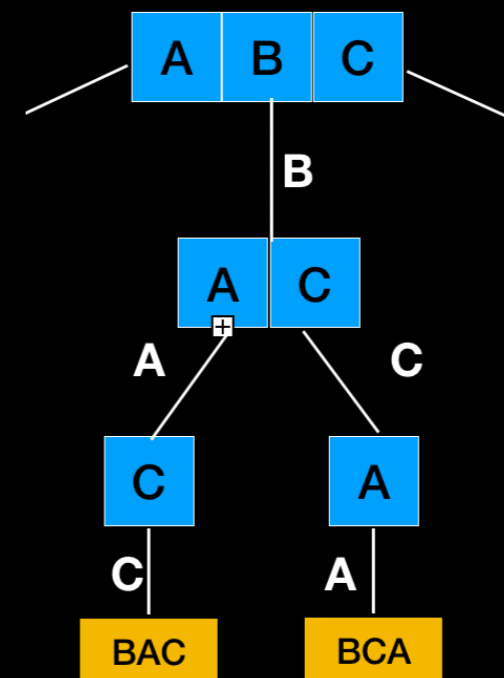
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```

i=1, first=0

BAC



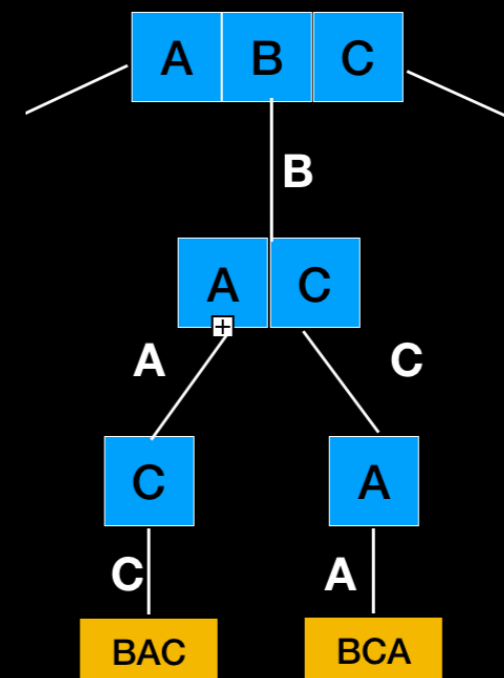
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```

i=1, first=0

BAC



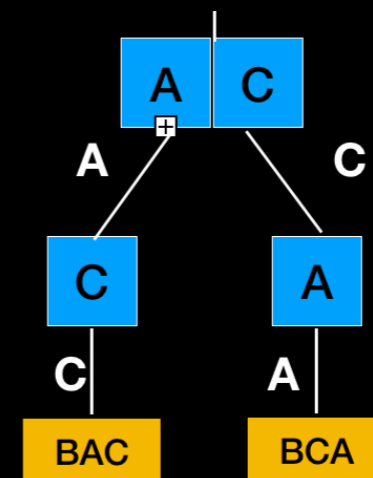
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```

i=1, first=1

BAC



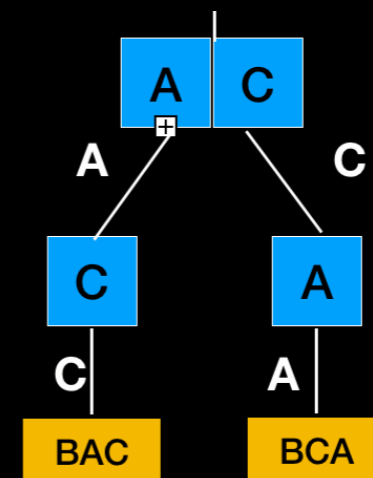
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i=1, first=1

BAC



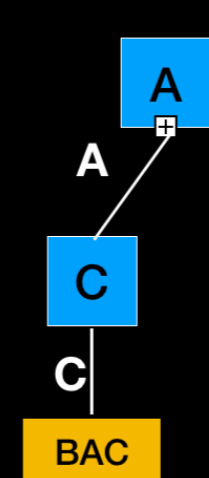
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```

i=1, first=1

BAC



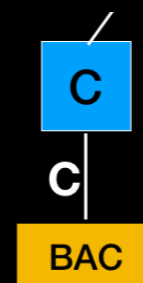
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            permuteStr(str, first+1, last);
            swap(str[first],str[i]); //restore first char
        }
    }
}

```

i=2, first=2

BAC



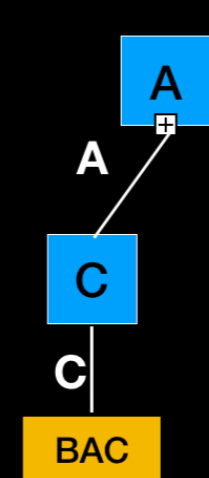

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    }
}

```

i=1, first=1

BAC



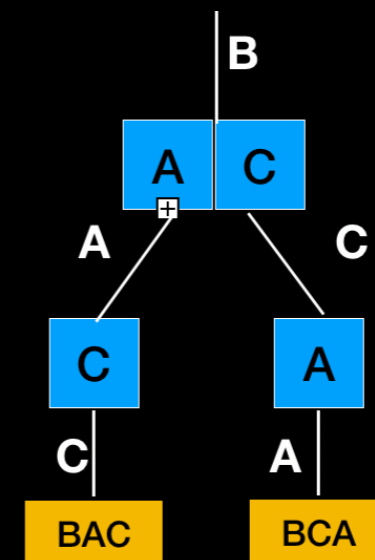
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}

```

i=2, first=1

BCA



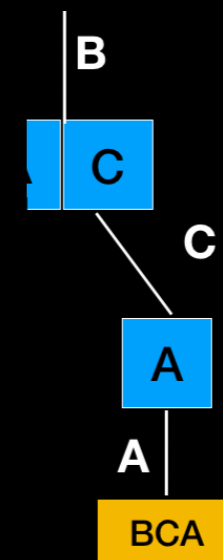
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```

i=2, first=1

BCA



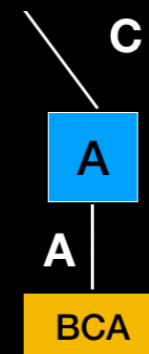
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```

i=2, first=2

BCA



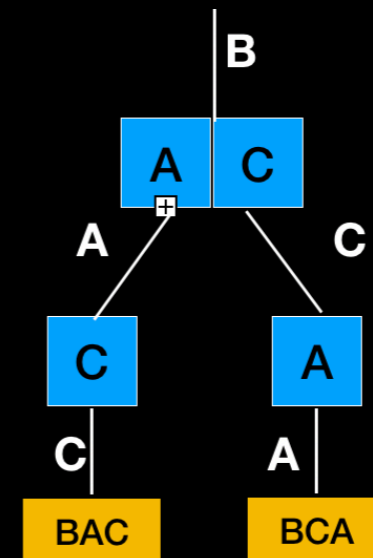
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```

i=2, first=1

BAC



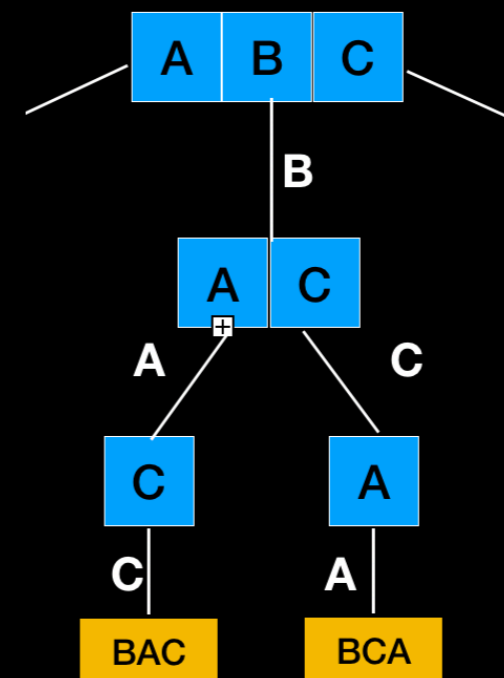
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        }
    }
}

```

i=1, first=0

ABC



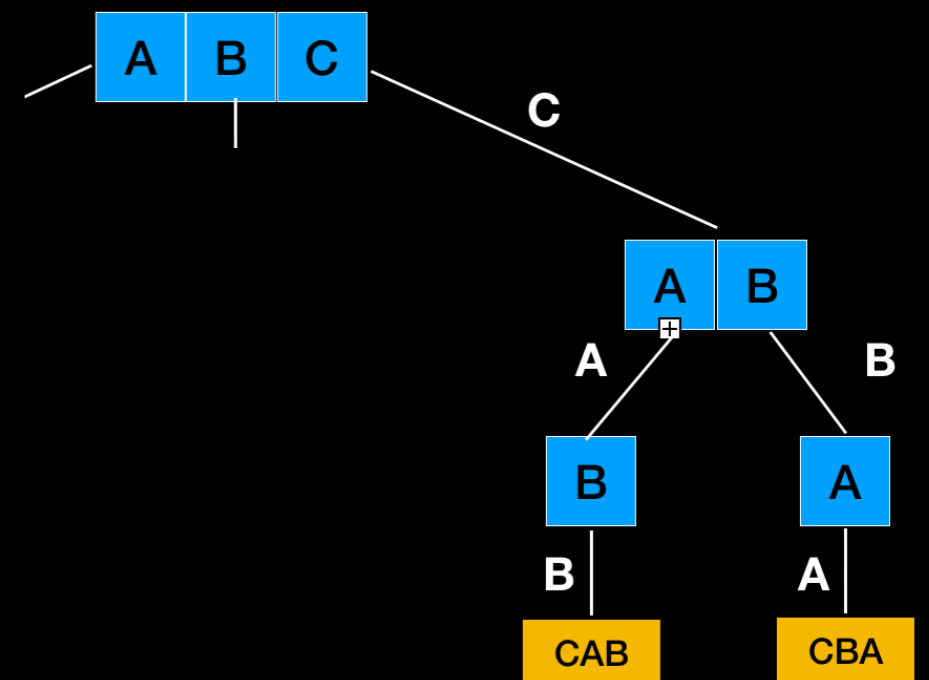
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        }
    }
}

```

i=2, first=0

ABC



Recursive Decision Tree

```
void exploreFrom(current_state, decisions_made) {  
    if (all decisions have been made) { //base case  
        output the result of the decisions we've made;  
    } else {  
        for (each decision we can make) {  
            → exploreFrom(result of making that decision,  
                           decisions_made + this_decision);  
        }  
    }  
}
```

Generally, if you can express a problem solution with a decision tree you can translate it into a recursive algorithm

Find Combinations

A B C D

Order does
Not matter!

{ }

A

A B

A B C

A B C D

B

A C

A B D

C

A D

A C D

D

B C

B C D

B D

C D

Find Combinations of size 2

A B C D

Order does
Not matter!

{ }

A

A B

A B C

A B C D

B

A C

A B D

C

A D

A C D

D

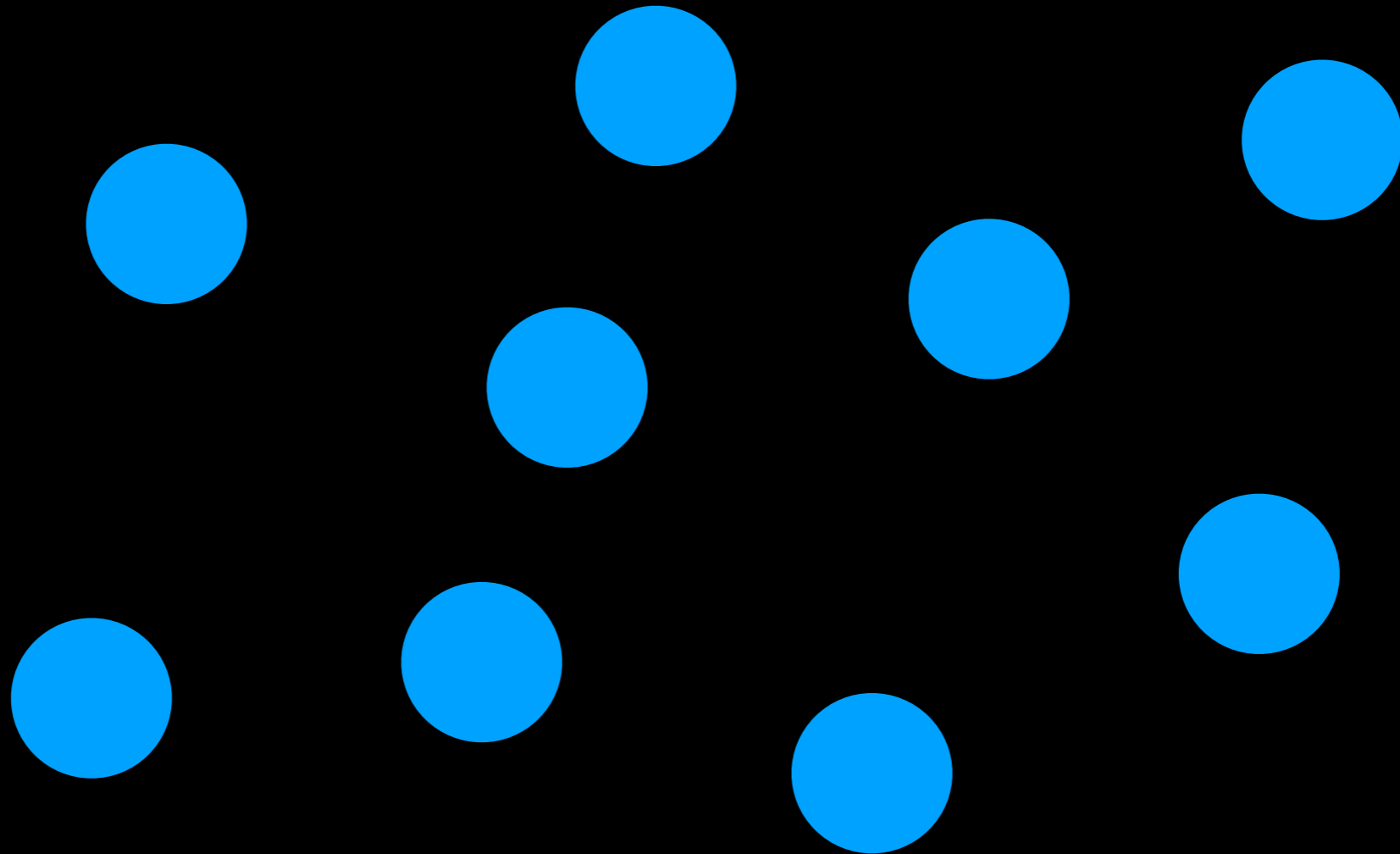
B C

B C D

B D

C D

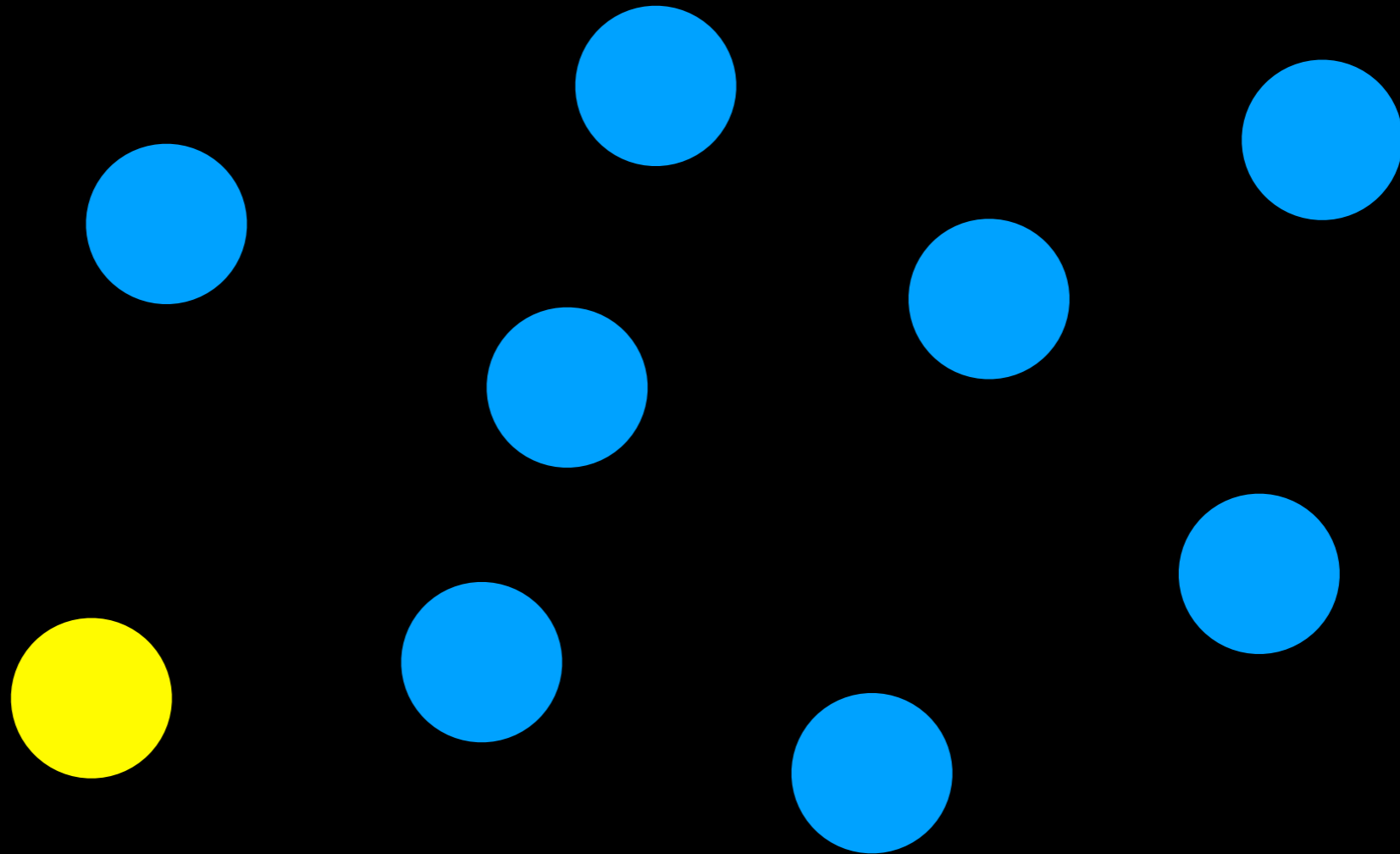
Combinations (n choose k)



One way to choose 5 out of 9 is to **Exclude 1** and **choose 5 out of 8**

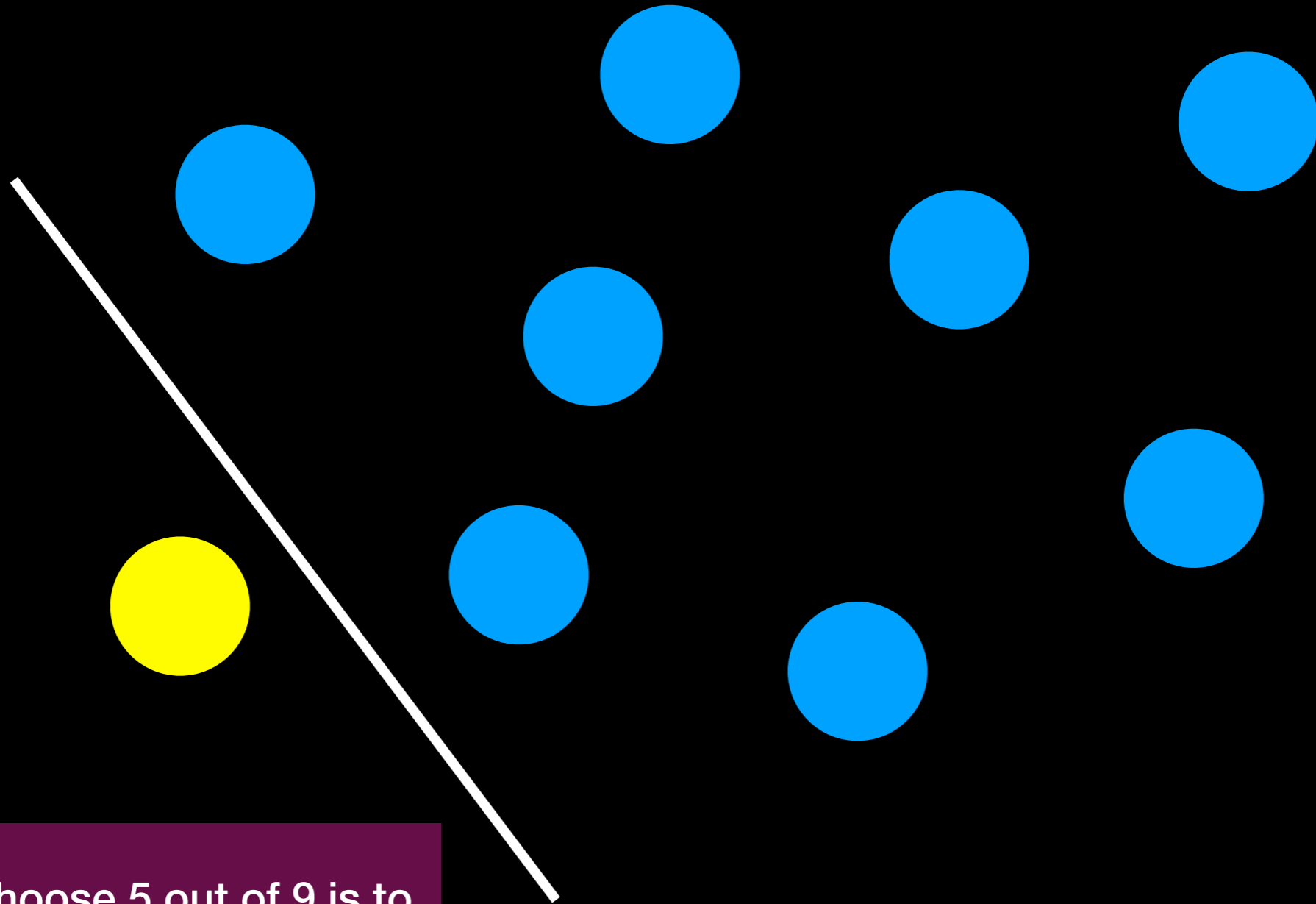
Start with toy problem
to make observation

Combinations (n choose k)



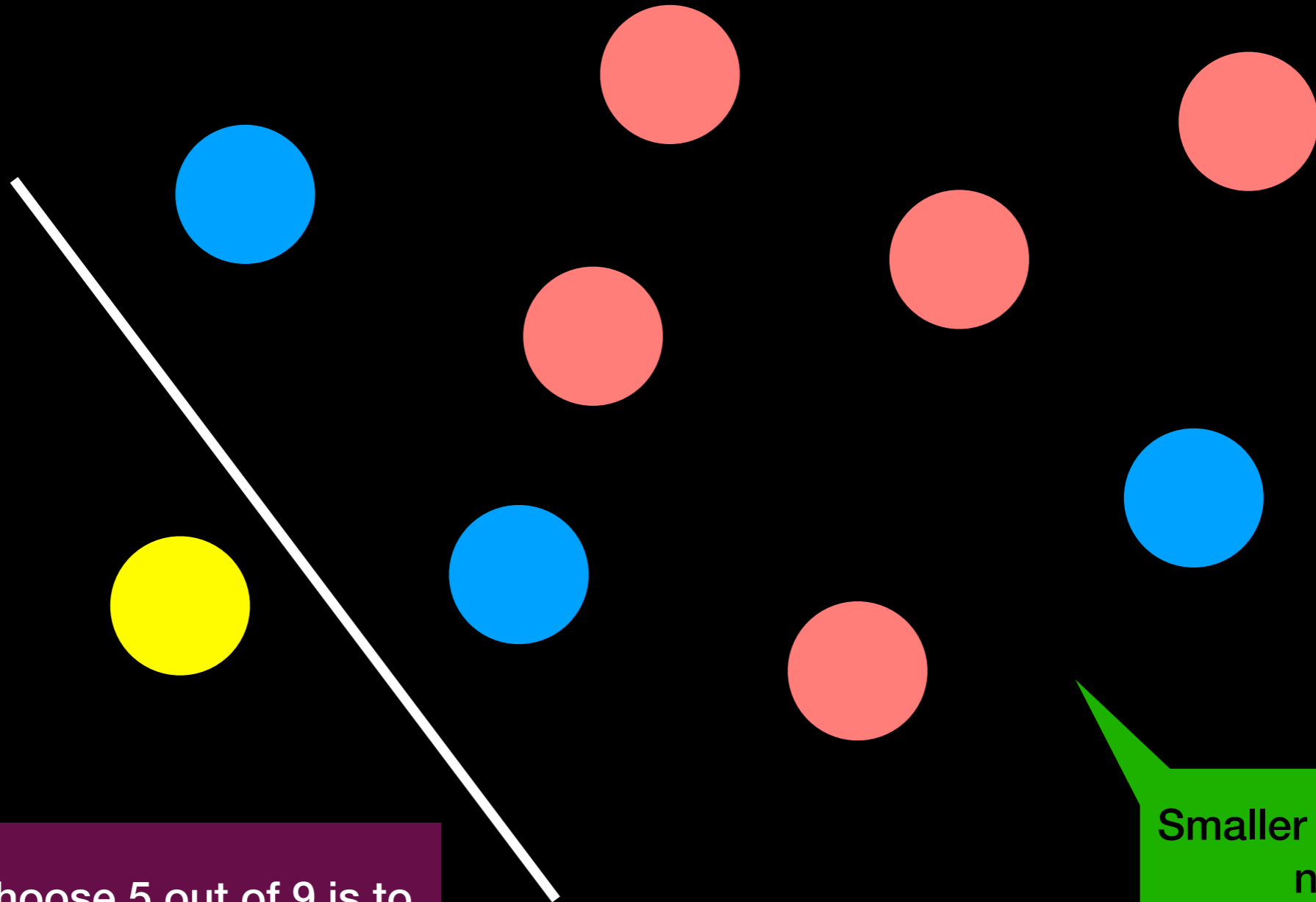
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Combinations (n choose k)



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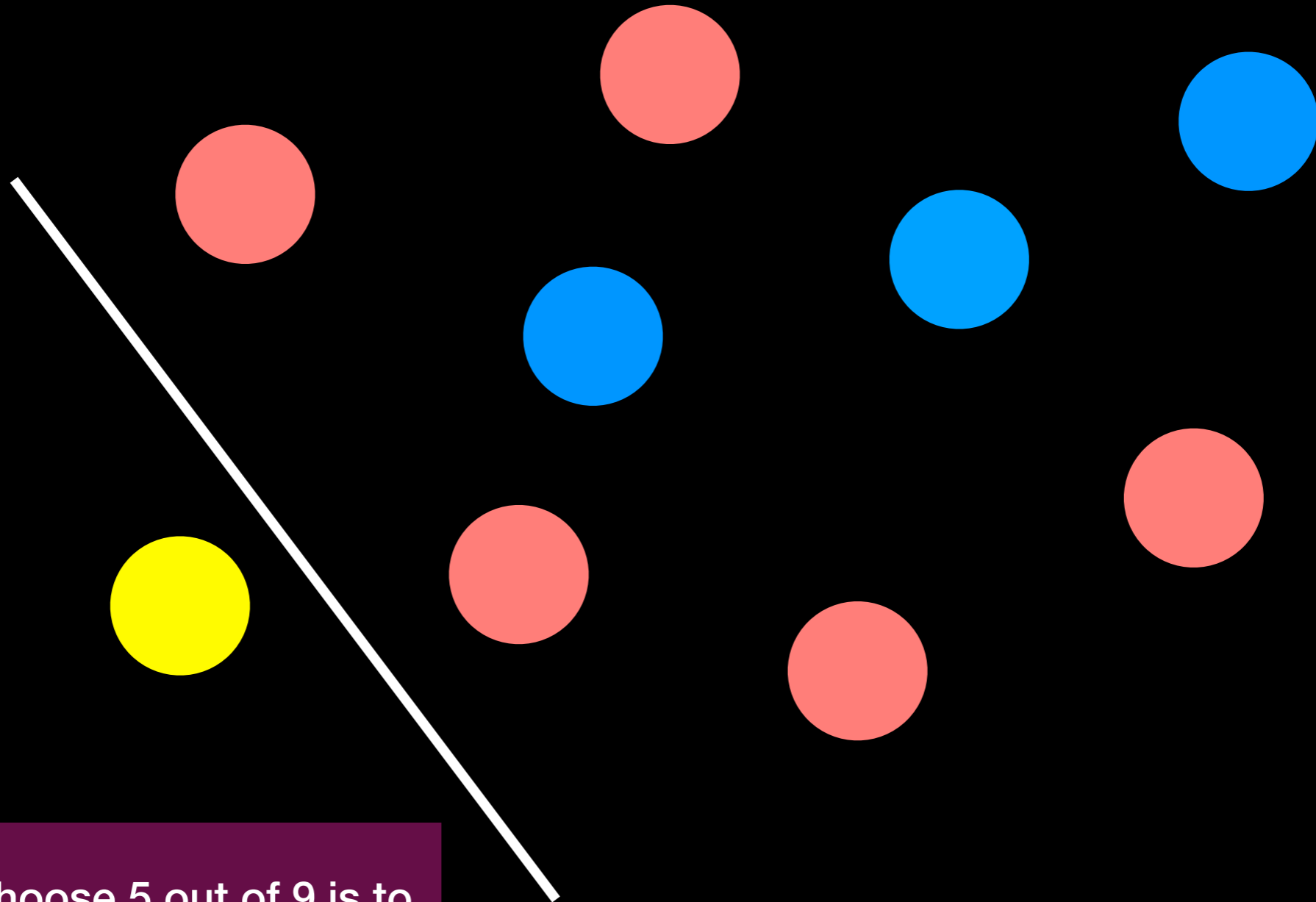
Combinations (n choose k)



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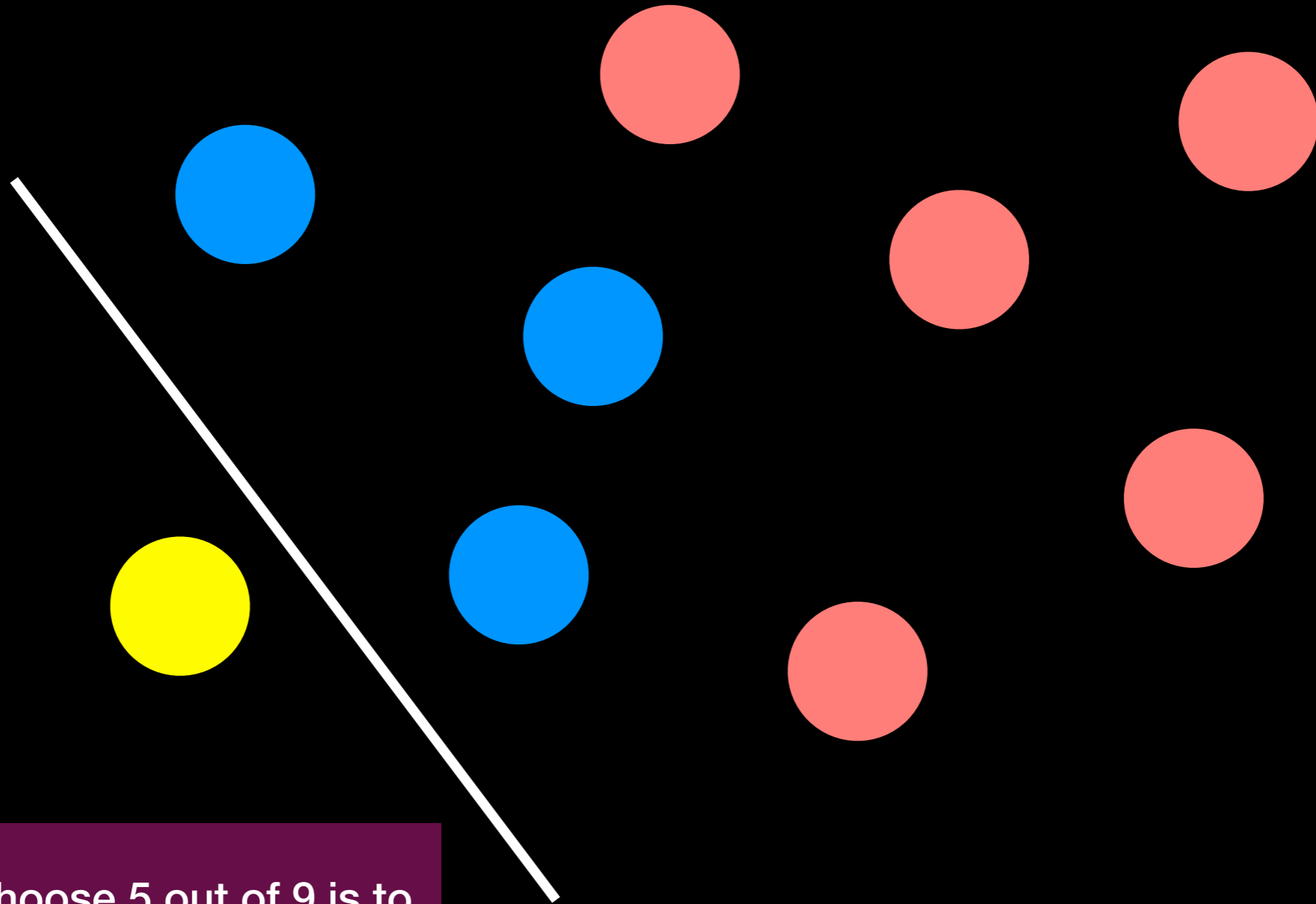
Smaller problem!
n-1

Combinations (n choose k)



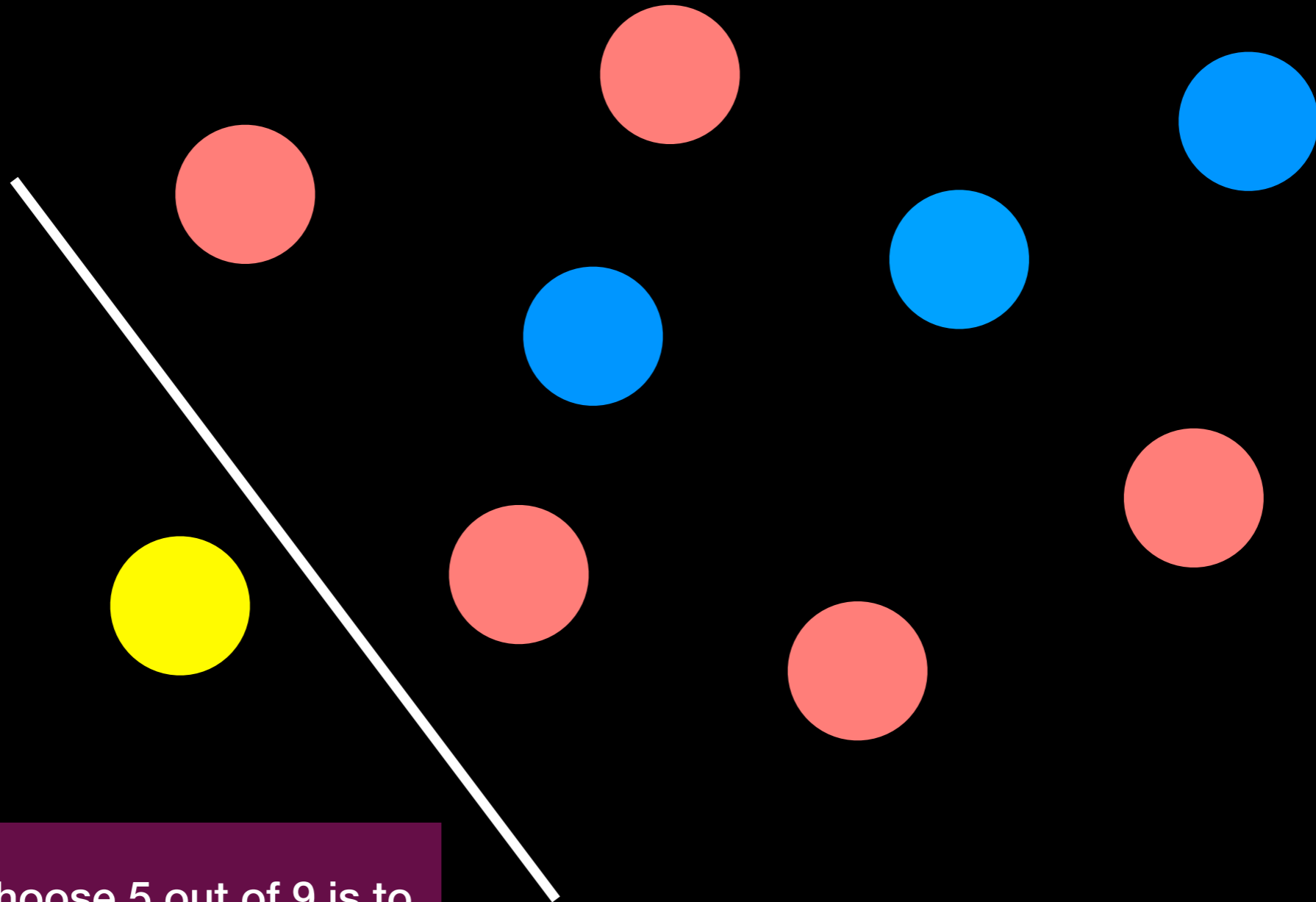
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Combinations (n choose k)



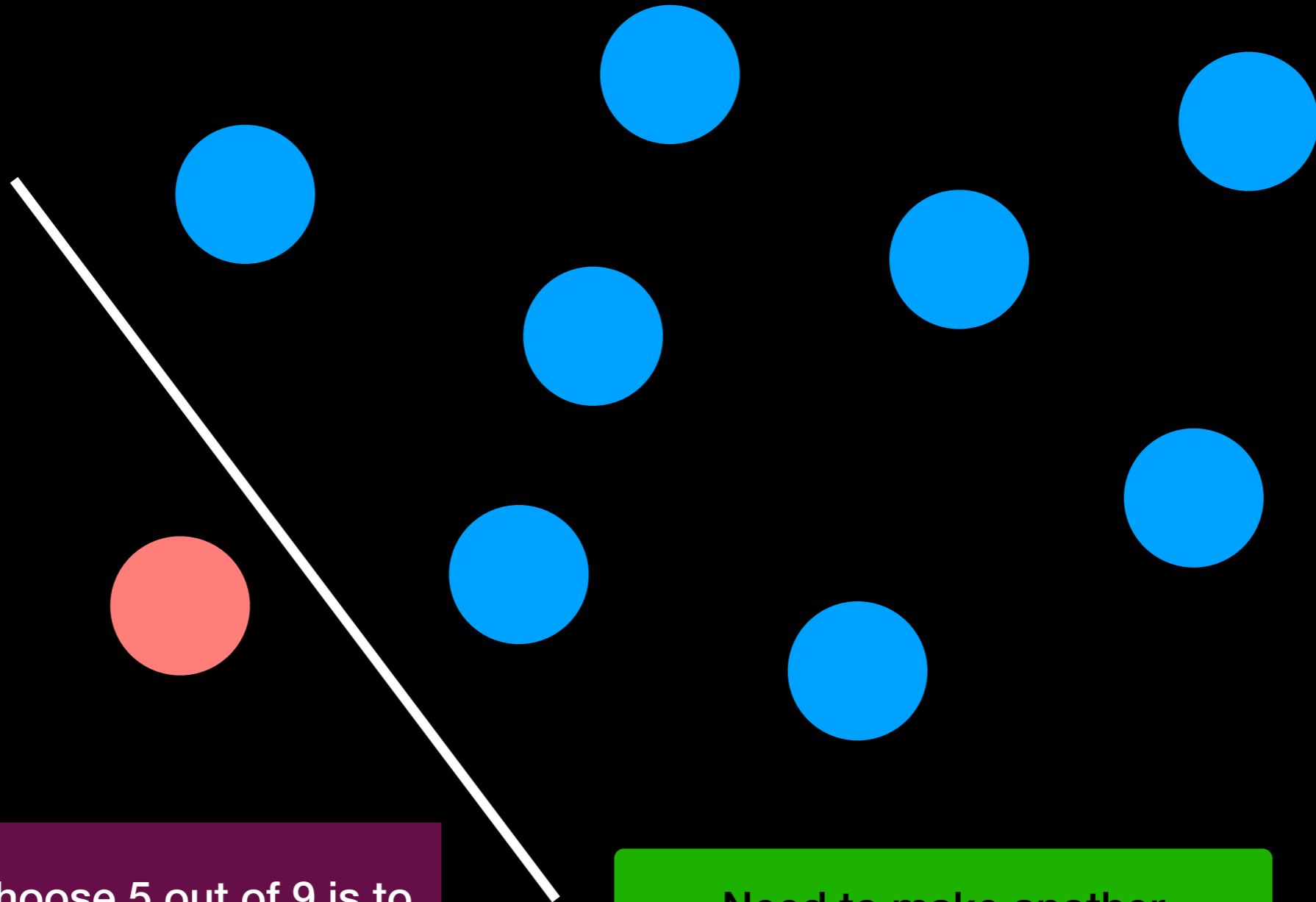
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Combinations (n choose k)



One way to choose 5 out of 9 is to
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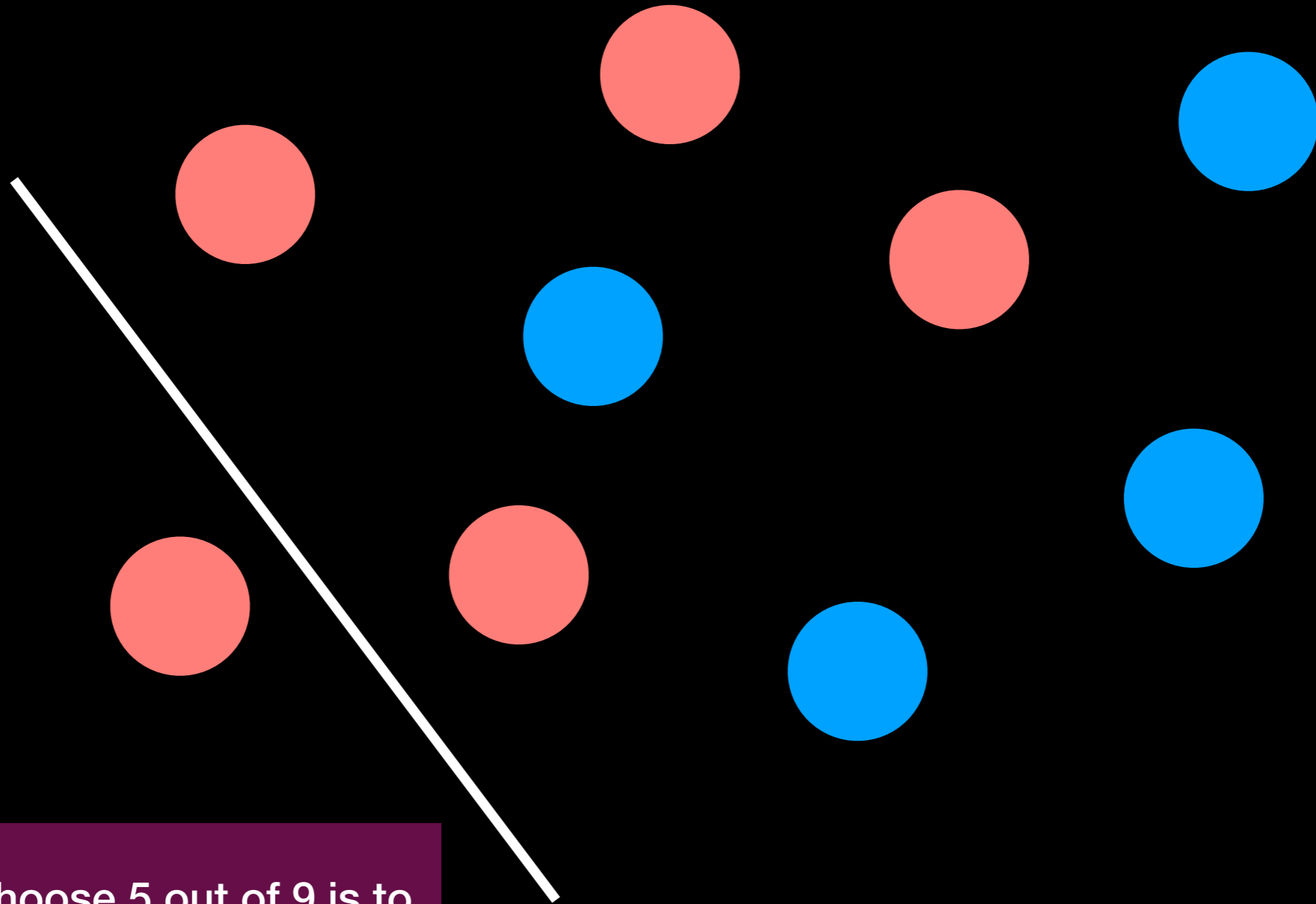
Combinations (n choose k)



One way to choose 5 out of 9 is to
Include 1 and choose 4 out of 8

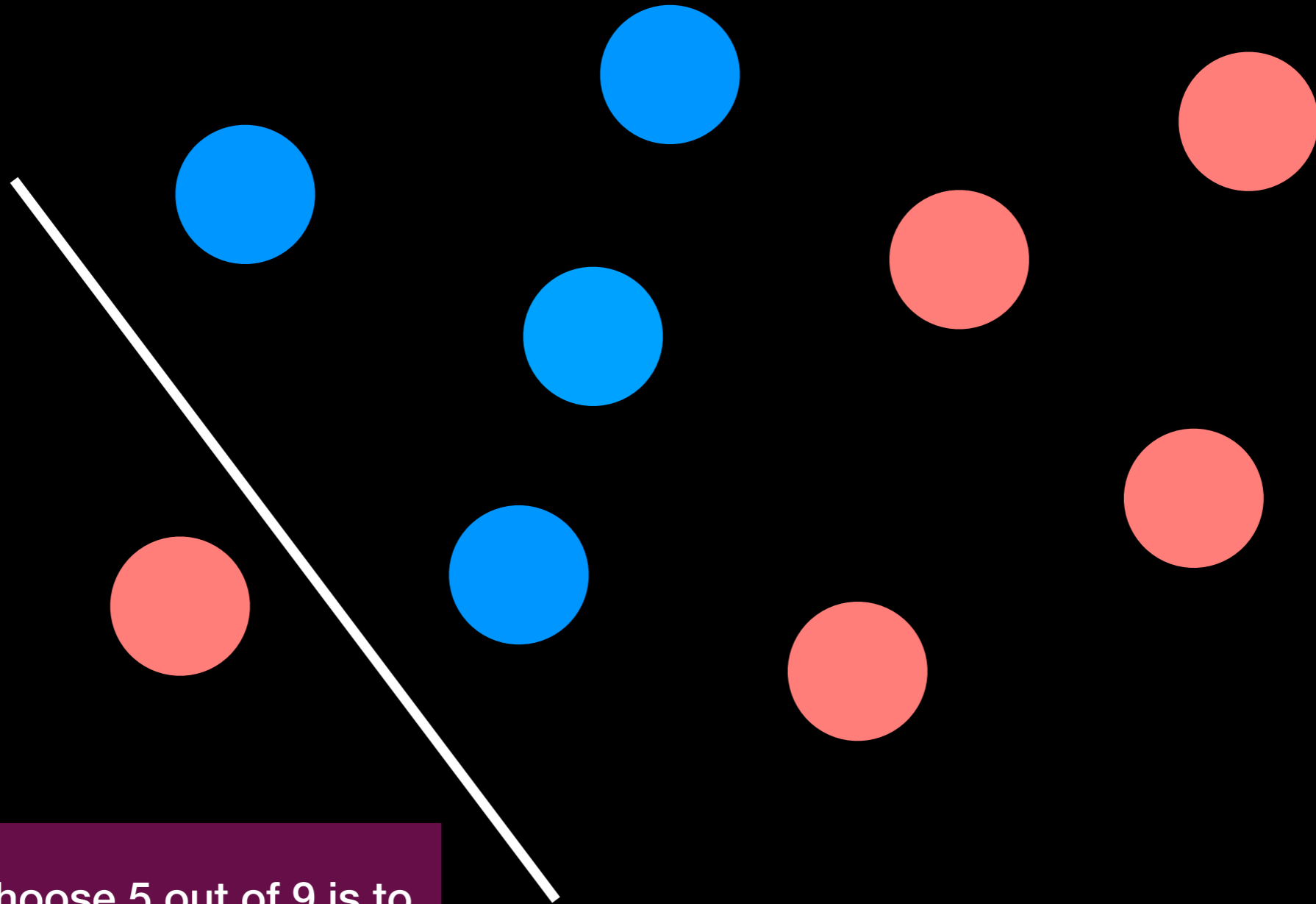
Need to make another
observation

Combinations (n choose k)



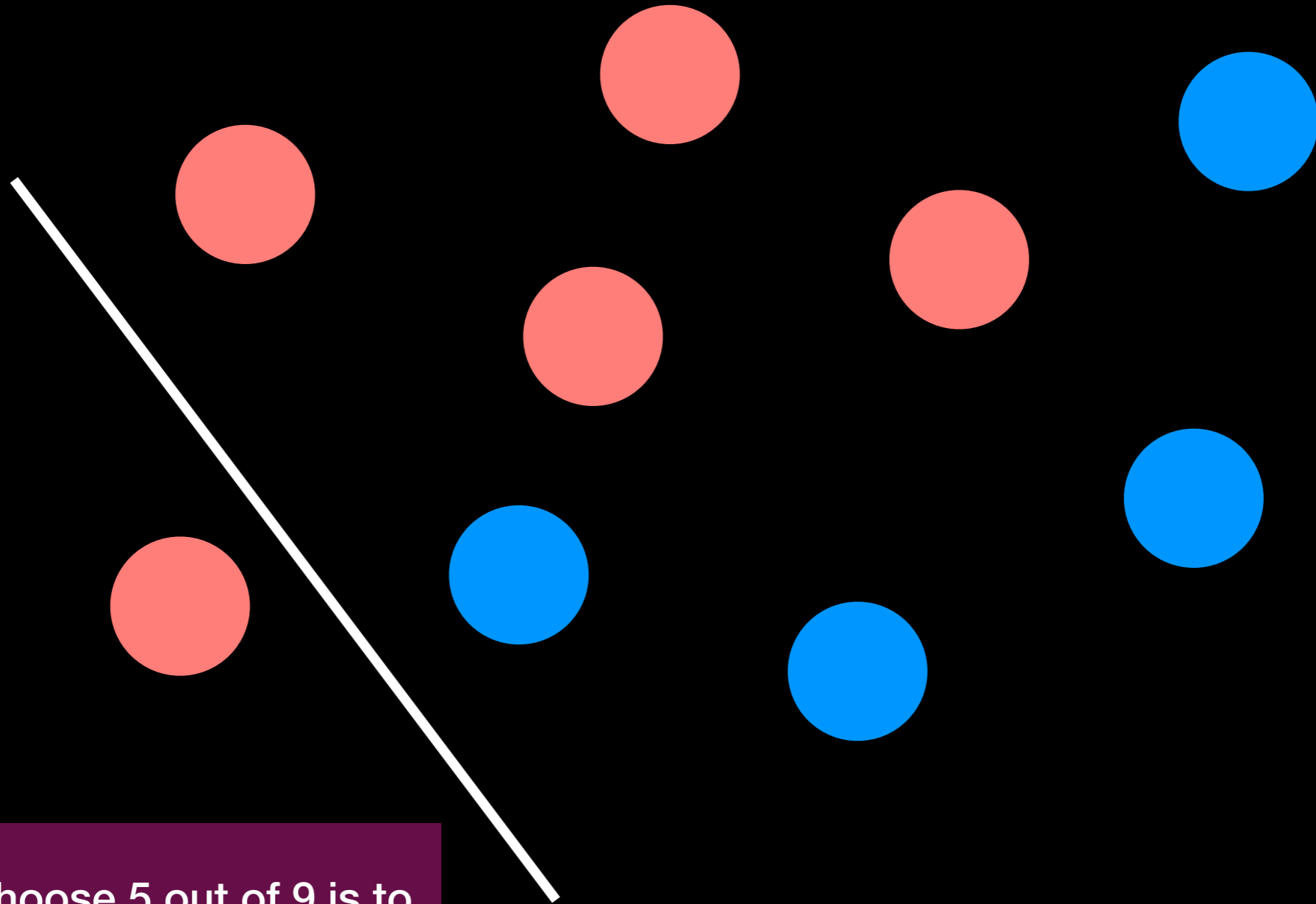
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Combinations (n choose k)



One way to choose 5 out of 9 is to
Include 1 and choose 4 out of 8

Combinations (n choose k)



One way to choose 5 out of 9 is to
Include 1 and choose 4 out of 8

Count Combinations

```
int countCombinations(int n, int k)
{
    if ( (k == 0) || (k == n) )
        return 1;
    else
        return countCombinations(n-1, k-1) +
               countCombinations(n-1, k);
}
```

Include one

Exclude one

Recursive algorithm for computing binomial coefficients

How can you be sure it works???

You come up with an algorithm

You implement it

You test it

How can you be sure it will ALWAYS work???

How can you be sure it works???

You come up with an algorithm

You implement it

You test it

How can you be sure it will ALWAYS work???

PROVE IT!!!

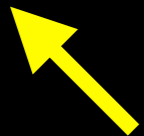
Recursion and Induction

Principle of Mathematical Induction:

Suppose you want to prove that a statement $P(n)$ about an integer n is true for every positive integer n .

To prove that $P(n)$ is true for all $n \geq 1$, do the following two steps:

- **Base Step:** Prove that $P(1)$ is true.
- **Inductive Step:** Let $k \geq 1$. Assume $P(k)$ is true, and prove that $P(k + 1)$ is also true.



Recursion and Induction

```
//a: nonzero real number, n: nonnegative integer
power(a, n)
{
  if (n = 0)
    return 1
  else
    return a * power(a, n - 1)
}
```

Prove by mathematical induction on n that the algorithm above is correct.

We will show $P(n)$ is true for all $n \geq 0$, where

$P(n)$: For all nonzero real numbers a , $\text{power}(a, n)$ correctly computes a^n .

Recursion and Induction

Base step: If $n = 0$, the first step of the algorithm tells us that $\text{power}(a, 0) = 1$. This is correct because $a^0 = 1$ for every nonzero real number a , so $P(0)$ is true.

Inductive step:

Let $k \geq 0$.

Inductive hypothesis: $\text{power}(a, k) = a^k$, for all $a \neq 0$.

We must show next that $\text{power}(a, k+1) = a^{k+1}$.

Since $k + 1 > 0$ the algorithm sets

$$\text{power}(a, k + 1) = a * \text{power}(a, k)$$

By inductive hypotheses $\text{power}(a, k) = a^k$

$$\text{so } \text{power}(a, k + 1) = a * \text{power}(a, k) = a * a^k = a^{k+1}$$

Exam Drill

Write a recursive function that returns true if the input string is a palindrome (same when reversed)

Exam Drill

Write a recursive function that returns true if the input string is a palindrome (same when reversed)

```
bool isPalindrome(string s)
{
    if(s.length() == 0 || s.length() == 1) //base case
        return true; //empty string or string of size 1 are palindrome
    if(s[0] == s[s.length()-1]) //if first and last char are same
        //check substring leaving out first and last character
        return isPalindrome(s.substr(1, s.length()-2));

    return false; //not palindrome
}
```

Exam Drill

Write a recursive function for the fibonacci numbers
where $f(n) = f(n-1) + f(n-2)$

Exam Drill

Write a recursive function for the fibonacci numbers where $f(n) = f(n-1) + f(n-2)$

```
int fib(int n)
{
    if (n <= 1)//base case
        return n;
    return fib(n-1) + fib(n-2);
}
```

Exam Drill

Write a recursive function to find the max value in an array of integers

Exam Drill

Write a recursive function to find the max value in an array of integers

```
int findMax(int* a, int index) {  
    if (index > 0)  
        return max(a[index], findMax(a, index-1));  
    else  
        return a[0];  
}
```

Exam Drill

Write a **recursive** function that finds a particular item in a **sorted** array (we will look at this algorithm soon)