Algorithm Efficiency (More formally)

Tiziana Ligorio

Today's Plan

Algorithm Efficiency

What is CSCI 235?

Programming => Software Analysis and Design

Expected professional and responsible attitude

Think like a Computer Scientist:

Design and maintain complex programs

- Software Engineering, Abstraction, OOP

Design and represent data and its management

- Abstract Data Types

Implement data representation and operations

- Data Structures
- Algorithms

Analyze Algorithms and their Efficiency

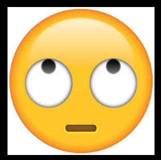
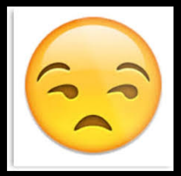
Algorithm Efficiency

Scenario 1

You are using an application and suddenly it stalls… whatever it is doing it's taking way too long…

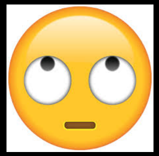
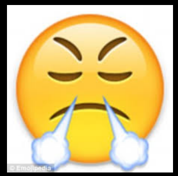
Scenario 1

You are using an application and suddenly it stalls… whatever it is doing it's taking way too long…

how "*long*" does that have to be for you to become ridiculously frustrated?    

Scenario 1

You are using an application and suddenly it stalls… whatever it is doing it's taking way too long…

how "*long*" does that have to be for you to become ridiculously frustrated?    

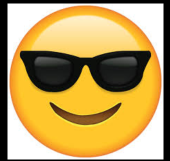
… probably not that long

Scenario 2

At your next super high-end job with the company/research-center of your dreams you are given a very difficult problem to solve.

You work hard on it, find a solution, code it up and it works!!!!

Proudly you present it the next day



but…

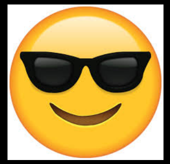
Given some new (large) input it keeps stalling…

Scenario 2

At your next super high-end job with the company/research-center of your dreams you are given a very difficult problem to solve.

You work hard on it, find a solution, code it up and it works!!!!

Proudly you present it the next day



but…

Given some new (large) input it keeps stalling…

Well… sorry but your solution is no good!!!

You need to have a means to estimate/predict the efficiency of your algorithms on **unknown input.**

What is a good solution?

How can we compare solutions to a problem? (Algorithms)

What is a good solution?

Correct

If it's not correct it is not a solution at all

What is a good solution?

Correct

Efficient

Time **Space**

What is a good solution?

Correct

Efficient

Time

Space

We are going to focus on time

How can we measure time efficiency?

How can we measure time efficiency?

Runtime?

Problems with actual runtime for comparison

What computer are you using?

- Runtime is highly sensitive to hardware

Problems with actual runtime for comparison

What computer are you using?

Runtime is highly sensitive to hardware

What implementation are you using?

Implementation details may affect runtime but are not reflective of algorithm efficiency

How should we measure execution time?

How should we measure execution time?

Constant

Number of “steps” or “operations” as a function of the size of the input

Variable

```cpp
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

What are the operations?
Let **n** be the number of nodes

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

What are the operations?
Let **n** be the number of nodes

1 node instantiation and assignment upon entering the loop

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

What are the operations?
Let **n** be the number of nodes

1 node instantiation and assignment upon entering the loop

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

pointer comparison

What are the operations?
Let **n** be the number of nodes

1 node instantiation and assignment upon entering the loop

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

call to `getNext()`

pointer comparison

What are the operations?
Let **n** be the number of nodes

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

1 node instantiation and assignment upon entering the loop

call to `getNext()`

pointer assignment

pointer comparison

What are the operations?
Let **n** be the number of nodes

1 node instantiation and assignment upon entering the loop

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

call to `getNext()`

pointer assignment

pointer comparison

call to `getItem()`

write to the console

What are the operations?
Let **n** be the number of nodes

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

K_0 1 node instantiation and assignment upon entering the loop

K_1 call to `getNext()`

K_2 pointer assignment

K_3 pointer comparison

K_4 call to `getItem()`

K_5 write to the console

What are the operations?
Let **n** be the number of nodes

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

K_0 1 node instantiation and assignment upon entering the loop

K_1 call to `getNext()`

K_2 pointer assignment

K_3 pointer comparison

K_4 call to `getItem()`

K_5 write to the console

$$\text{Operations} = K_0 + n(K_1+K_2+K_3+K_4+K_5)$$

What are the operations?
Let **n** be the number of nodes

```
template<typename ItemType>
void List<ItemType>::traverse()
{
    for(Node<ItemType>*  ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        cout << ptr->getItem() << endl;
    }
}
```

K_0 1 node instantiation and assignment upon entering the loop

K_1 call to `getNext()`

K_2 pointer assignment

K_3 pointer comparison

K_4 call to `getItem()`

K_5 write to the console

$$\text{Operations} = K_0 + nK_6$$

Lecture Activity

Identify the steps and write down an expression for execution time

```
bool linearSearch(const string& str, char ch)
{

    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

Lecture Activity

Identify the steps and write down an expression for execution time

```
bool linearSearch(const string& str, char ch)
{

    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

Was this tricky?

n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
     // 1 int assignment upon entering loop
    for (int i = 0; i < str.length(); i++)
    { // call to length() and increment
        if (str[i] == ch) { // Comparisons
            return true; //return operation, maybe
        }
    }
    return false; //return operation, maybe
}
```

n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
     // 1 int assignment upon entering loop
    for (int i = 0; i < str.length(); i++)
    { // call to length() and increment
        if (str[i] == ch) { // Comparisons
            return true; //return operation, maybe
        }
    }
    return false; //return operation, maybe
}
```

Maybe stop in the middle

Maybe stop at end of loop

n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

In the
WORST CASE

Execution completes in **at most:**

$k_0 n + k_1$ operations

Types of Analysis

Best case analysis: running time under best input (e.g., in linear search item we are looking for is the first) - not reflective of overall performance)

Average case analysis: assumes equal probability of input (usually **not** the case)

Expected case analysis: assumes probability of occurrence of input is known or can be estimated, and if it were possible may be too expensive

Worst case analysis: running time under worst input, gives upper bound, it can't get worse, good for sleeping well at night!

n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

Execution completes in **at most:**

$k_0 \mathbf{n} + k_1$ operations

Some constant number of operations repeated inside the loop

Some constant number of operations performed outside the loop

n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

Execution completes in **at most:**

$k_0 \mathbf{n} + k_1$ operations

The number of times the loop is repeated, i.e. the size of `str`

Some constant number of operations repeated inside the loop

Some constant number of operations performed outside the loop

Observation

Don't need to explicitly compute the constants k_i

4**n** + 1000

n + 137

Dominant term is sufficient to explain overall behavior (in this case linear)

Big-O Notation

Ignores everything except **dominant term**

Examples:

Notation: describes the overall behavior

$T(n) = 4n + 4 = O(n)$

$T(n) = 164n + 35 = O(n)$

$T(n) = n^2 + 35n + 5 = O(n^2)$

$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$

$T(n) = 2^n + 5 = O(2^n)$

Big-O Notation

T(n) is the running time

n is the size of the input

Ignores everything except **dominant term**

Examples:

$T(n) = 4n + 4 = O(n)$

Notation: describes the overall behavior

$T(n) = 164n + 35 = O(n)$

$T(n) = n^2 + 35n + 5 = O(n^2)$

$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$

$T(n) = 2^n + 5 = O(2^n)$

Big-O Notation

Ignores everything except **dominant term**

Examples:

$T(n) = 4n + 4 = O(n)$

$T(n) = 164n + 35 = O(n)$

$T(n) = n^2 + 35n + 5 = O(n^2)$

$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$

$T(n) = 2^n + 5 = O(2^n)$

Big-O describes the overall behavior

Let *T(n)* be the ***running time*** of an algorithm measured as number of operations given **input of size n**.

$T(n)$ is $O(f(n))$

if it grows **no faster** than *f(n)*

Big-O Notation

But
$164n+35 > n$

Ignores everything except **dominant term**

Examples:

$T(n) = 4n + 4 = O(n)$

$T(n) = 164n + 35 = O(n)$

$T(n) = n^2 + 35n + 5 = O(n^2)$

$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$

$T(n) = 2^n + 5 = O(2^n)$

Big-O describes the overall behavior

Let $T(n)$ be the ***running time*** of an algorithm measured as number of operations given **input of size n**.

$T(n)$ is $O(f(n))$

if it grows **no faster** than $f(n)$

Big-O Notation

Ignores everything except **dominant term**

Examples:

$T(n) = 4n + 4 = O(n)$

$T(n) = 164n + 35 = O(n)$

$T(n) = n^2 + 35n + 5 = O(n^2)$

$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$

$T(n) = 2^n + 5 = O(2^n)$

Notation: describes the overall behavior

More formally:

T(n)* is *O(f(n))

if there exist constants k and n_0

such that for all $n \geq n_0$

$T(n) \leq kf(n)$

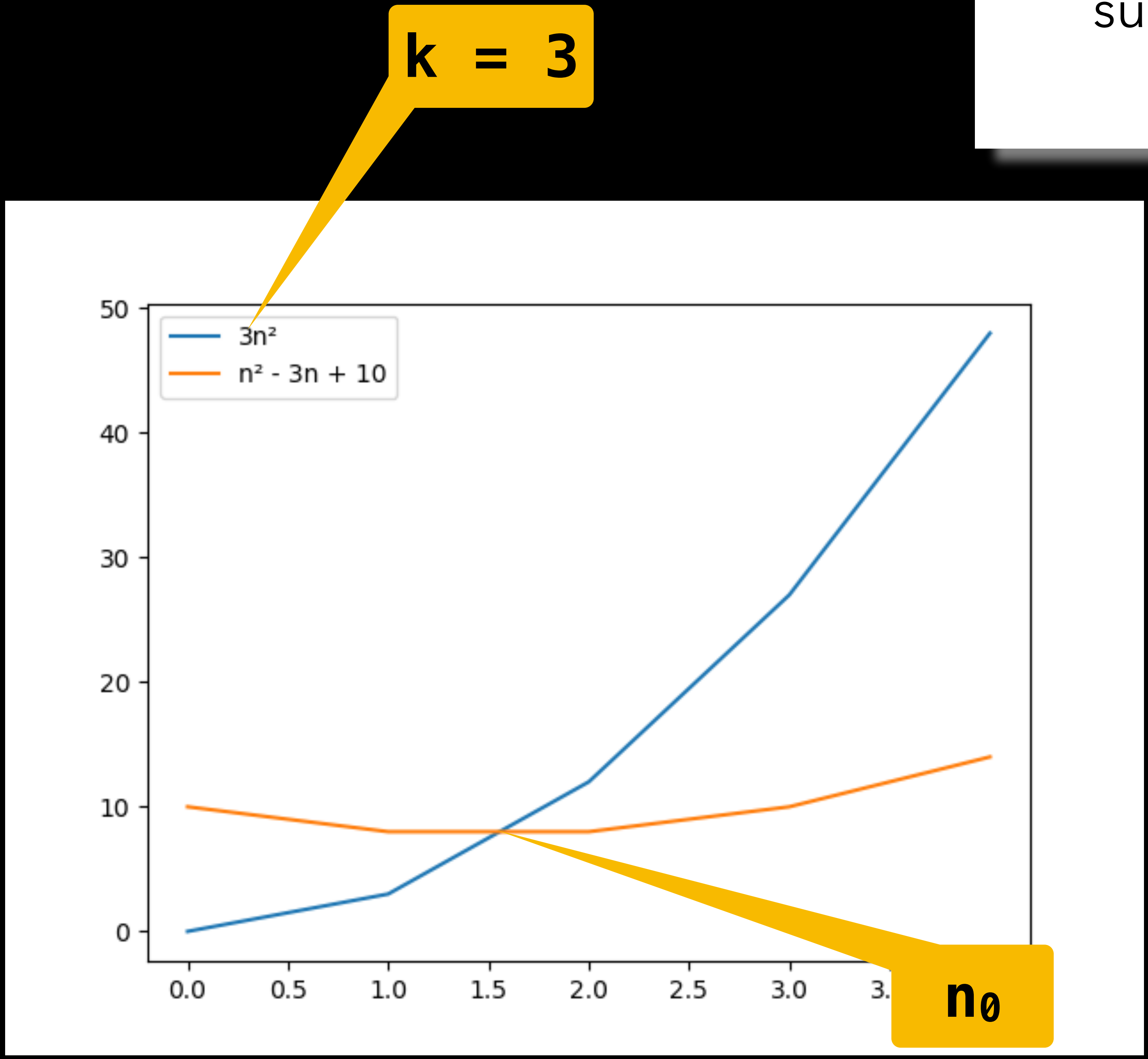
More formally:

$T(n)$ is $O(f(n))$

if there exist constants k and n_0

such that for all $n \geq n_0$,

$T(n) \leq kf(n)$



$T(n) = n^2 - 3n + 10$

$T(n)$ is $O(n^2)$

For k=3 and n≥1.5

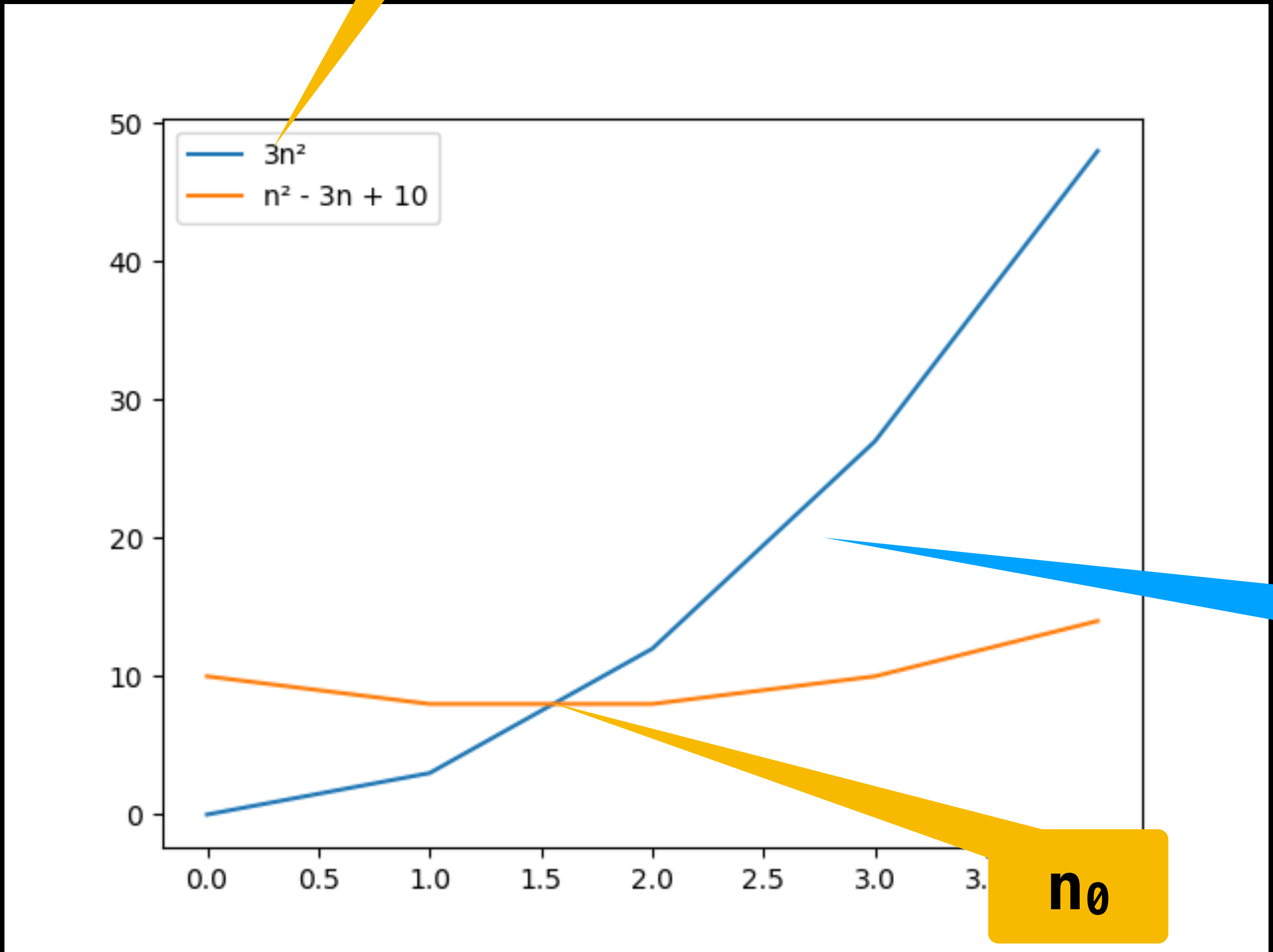
More formally:

$T(n)$ is $O(f(n))$

if there exist constants k and n_0
such that for all $n \geq n_0$,

$T(n) \leq kf(n)$

k = 3



n_0

$T(n) = n^2 - 3n + 10$
$T(n)$ is $O(n^2)$
For k=3 and $n \geq 1.5$

This is why we can look at **dominant term only** to explain behavior

Big-O describes the overall growth rate of an algorithms for **large n**

Proving Big-O Relationship

Apply definition of Big-O to prove that $T(n)$ is $O(f(n))$ for particular functions T and f

Do so by choosing k and n_0 s.t. for all $n \geq n_0$, $T(n) \leq kf(n)$

Proving Big-O Relationship

Example:

Suppose $T(n) = (n+1)^2$

We can say that $T(n)$ is $O(n^2)$

To prove it must find k and n_0 s.t. for all $n \geq n_0$,

$(n+1)^2 \leq kn^2$

Proving Big-O Relationship

Example:

Suppose $T(n) = (n+1)^2$

We can say that $T(n)$ is $O(n^2)$

To prove it must find k and n_0 s.t. for all $n \geq n_0$,

$(n+1)^2 \leq kn^2$

Expand $(n+1)^2 = n^2 + 2n + 1$

Observe that, as long as $n \geq 1$, $n \leq n^2$ and $1 \leq n^2$

Proving Big-O Relationship

Example:

Suppose $T(n) = (n+1)^2$

We can say that $T(n)$ is $O(n^2)$

To prove it must find k and n_0 s.t. for all $n \geq n_0$,

$(n+1)^2 \leq kn^2$

Expand $(n+1)^2 = n^2 + 2n + 1$

Observe that, as long as $n \geq 1$, $n \leq n^2$ and $1 \leq n^2$

Thus if we choose $n_0 = 1$ and $k = 4$ we have

$$\underbrace{n^2 + 2n + 1}_{T(n)} \leq n^2 + 2n^2 + n^2 = \underbrace{4}_{k}\underbrace{n^2}_{f(n)}$$

Proving Big-O Relationship

Example:

Suppose $T(n) = (n+1)^2$

We can say that $T(n)$ is $O(n^2)$

To prove it must find k and n_0 s.t. for all $n \geq n_0$,

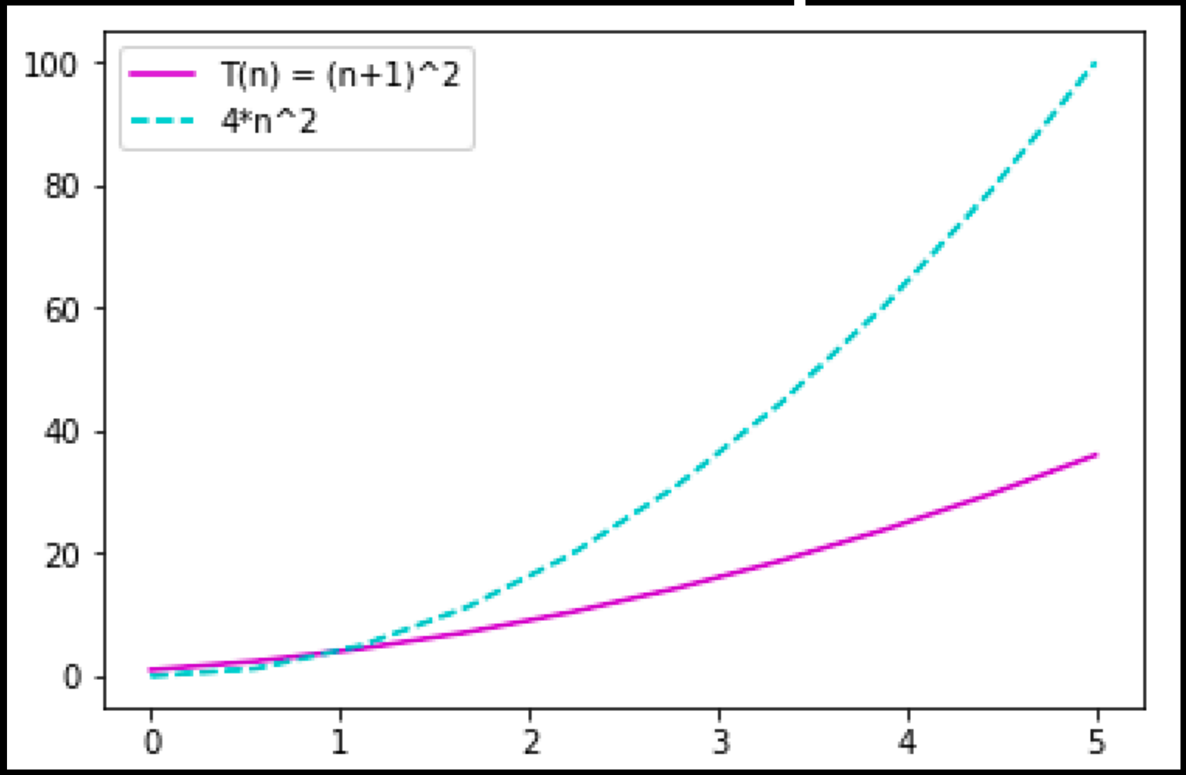
$(n+1)^2 \leq kn^2$

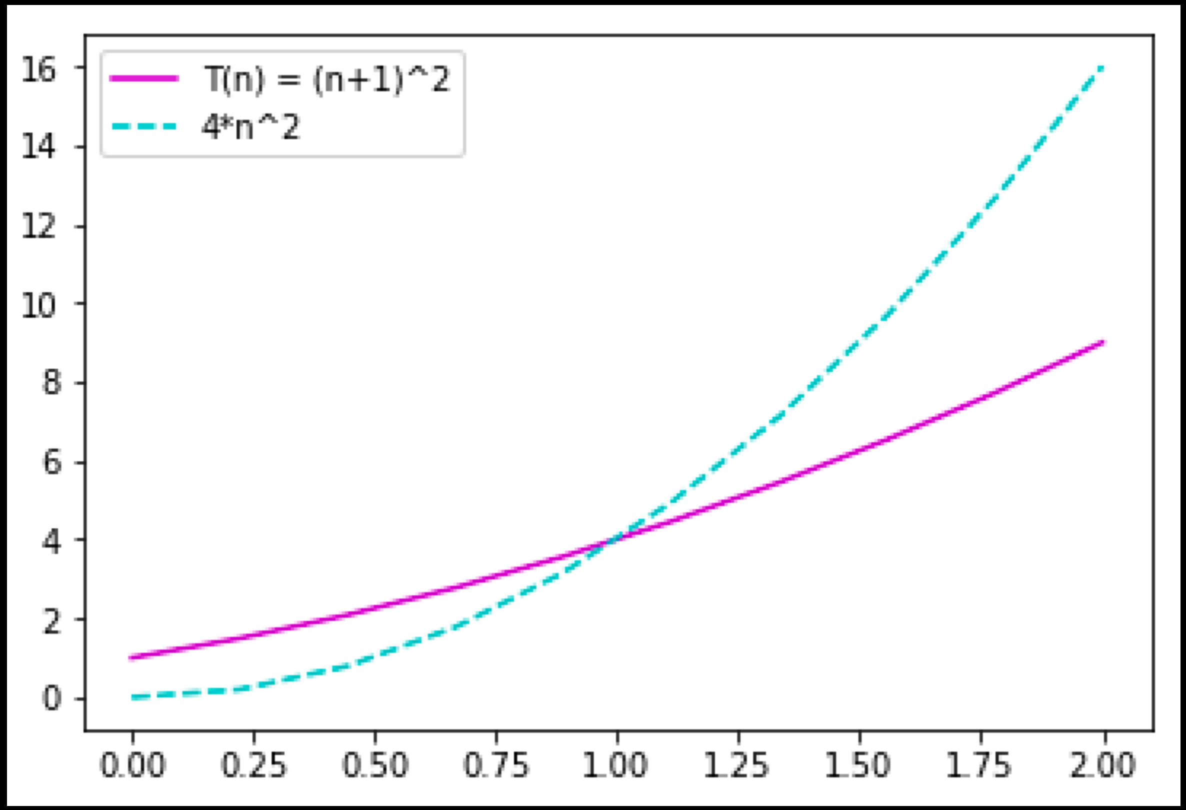
Expand $(n+1)^2 = n^2 + 2n + 1$

Observe that, as long as $n \geq 1$, $n \leq n^2$ and $1 \leq n^2$

Thus if we choose $n_0 = 1$ and $k = 4$ we have

$n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 = 4n^2$ ✓





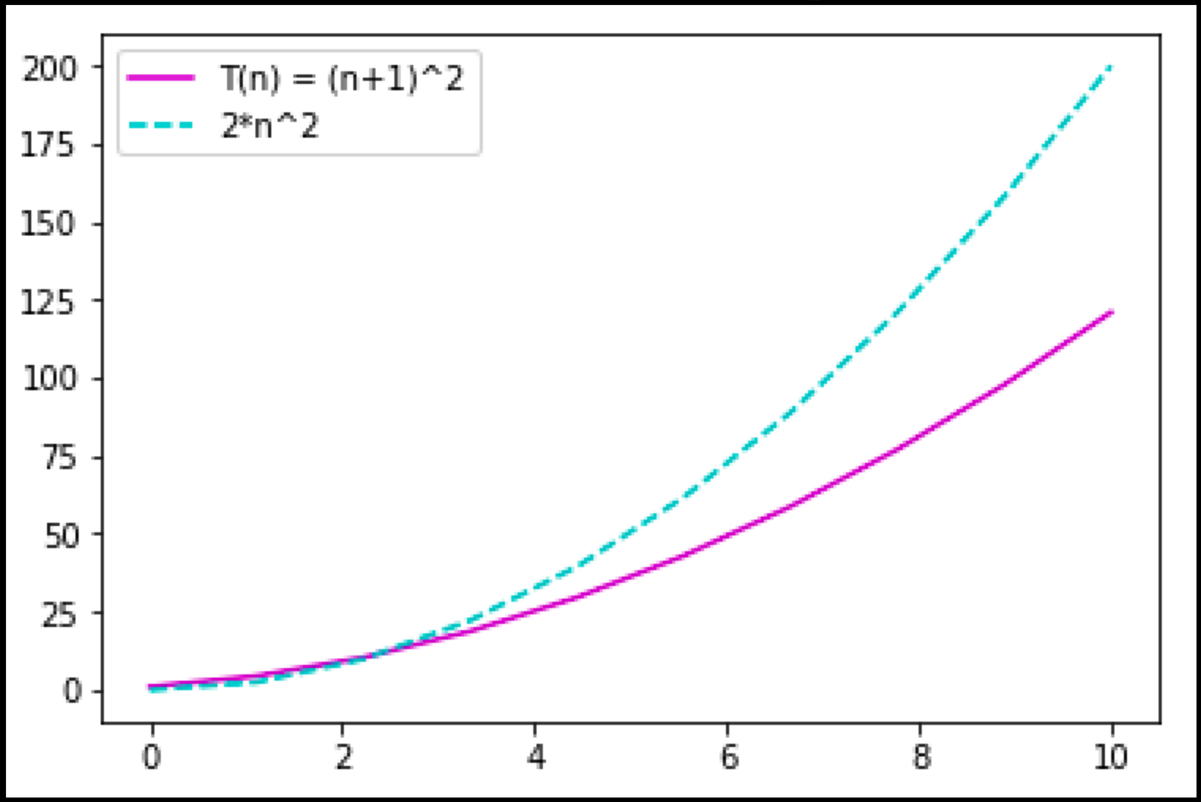
Proving Big-O Relationship

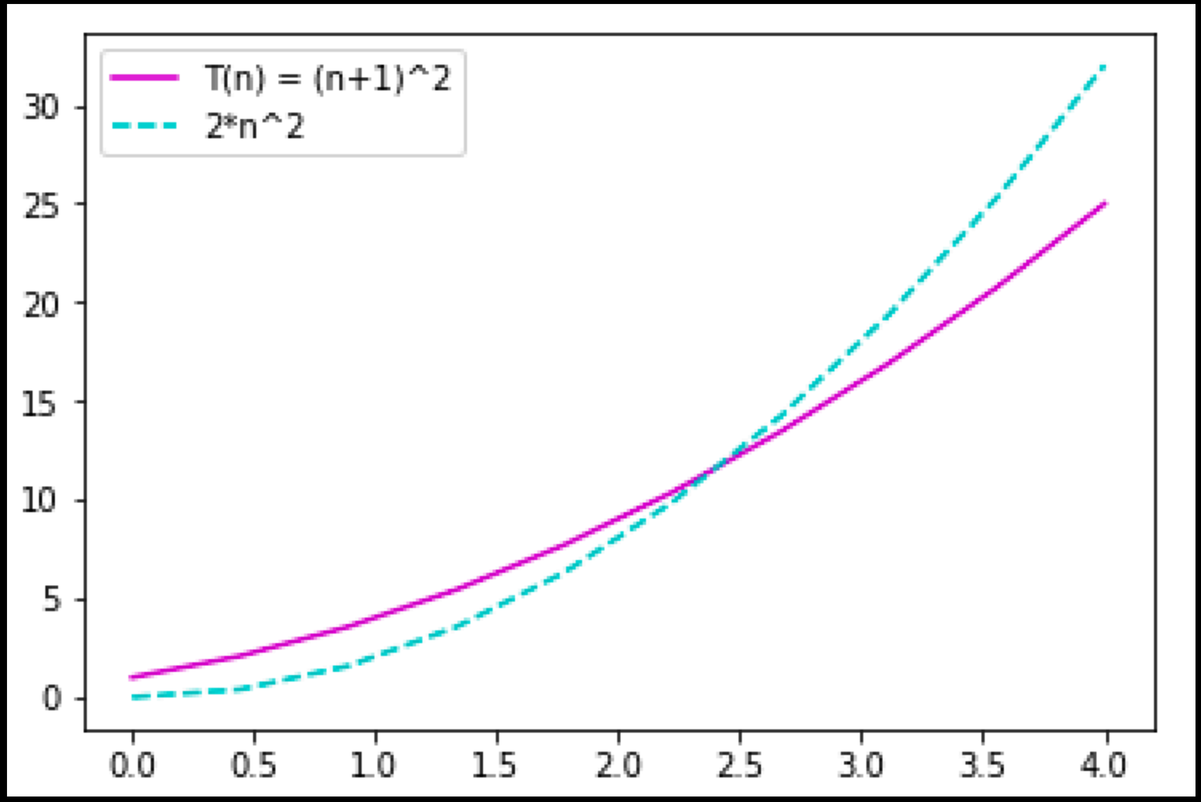
Not Unique:

Could also choose $n_0 = 3$ and $k = 2$ because

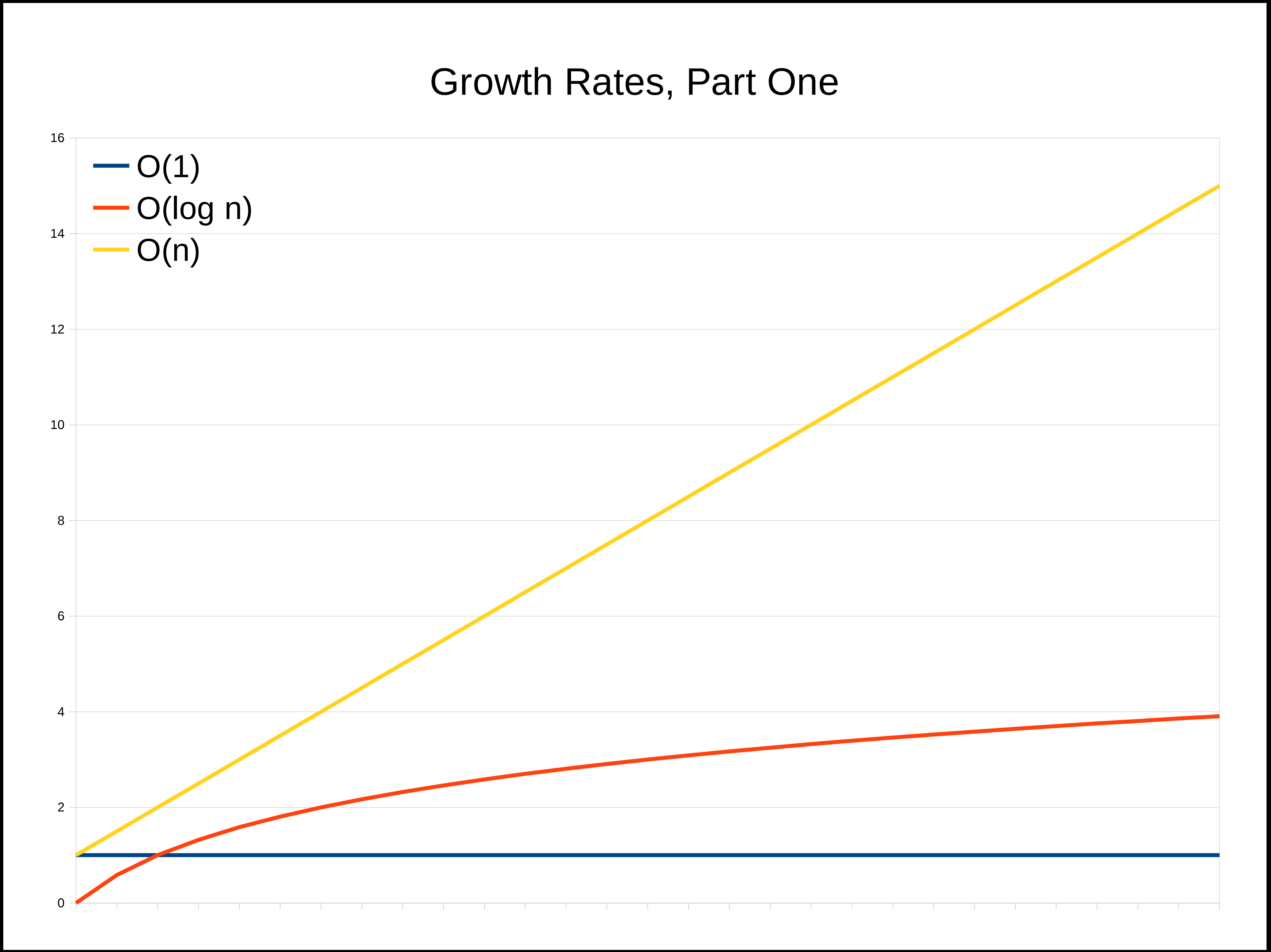
$(n+1)^2 \leq 2n^2$ for all $n \geq 3$

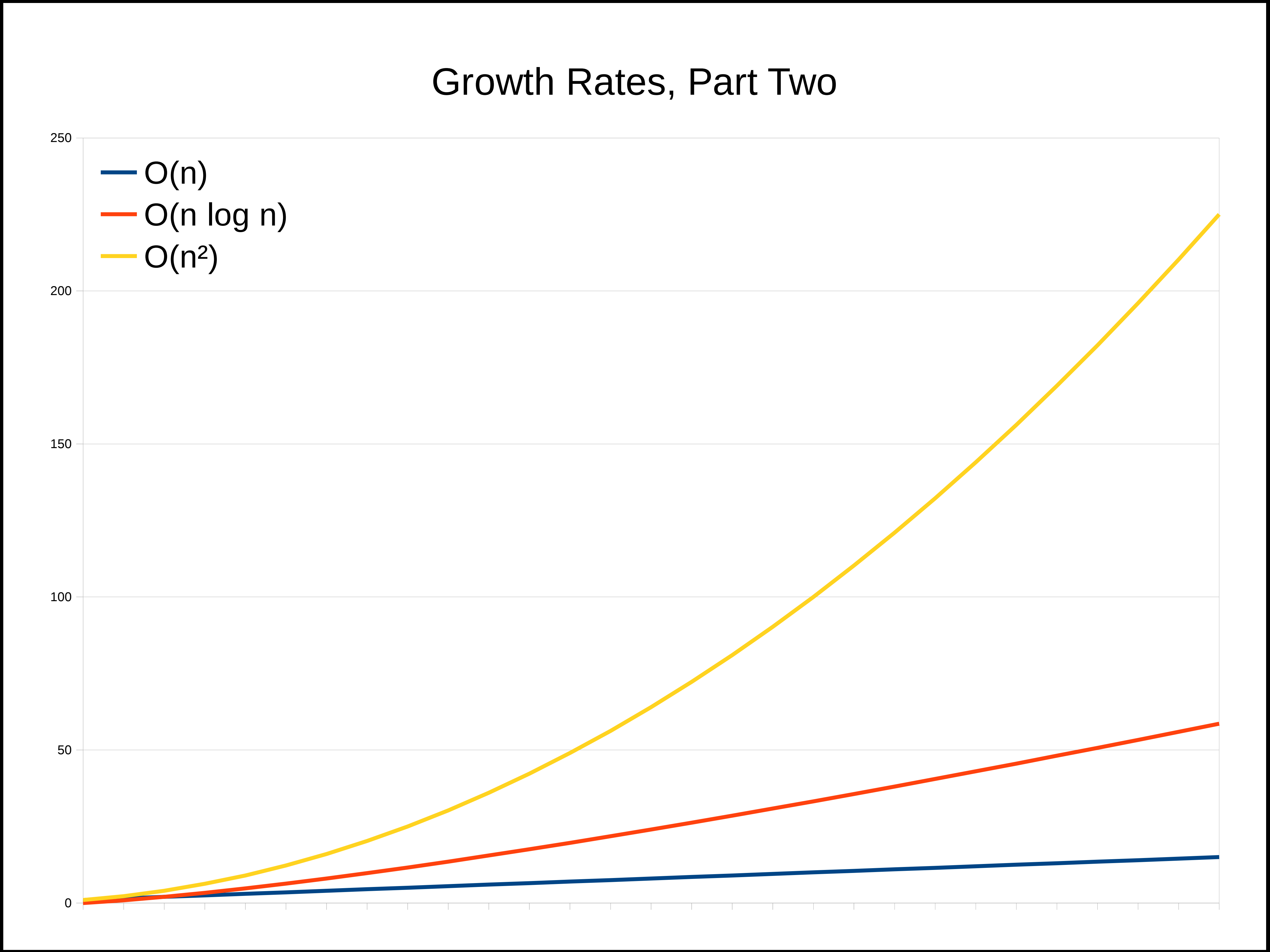
For proof one is enough

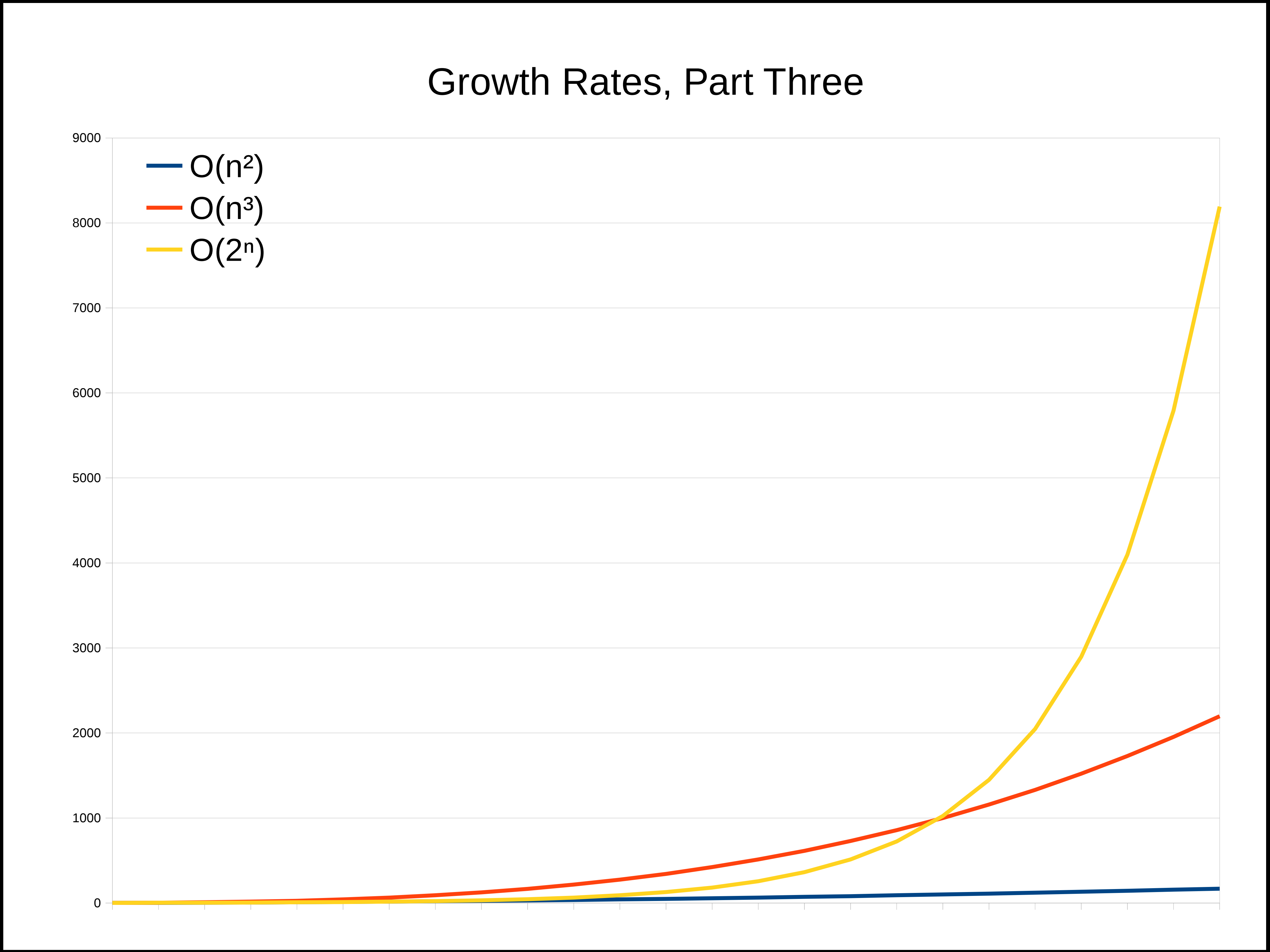


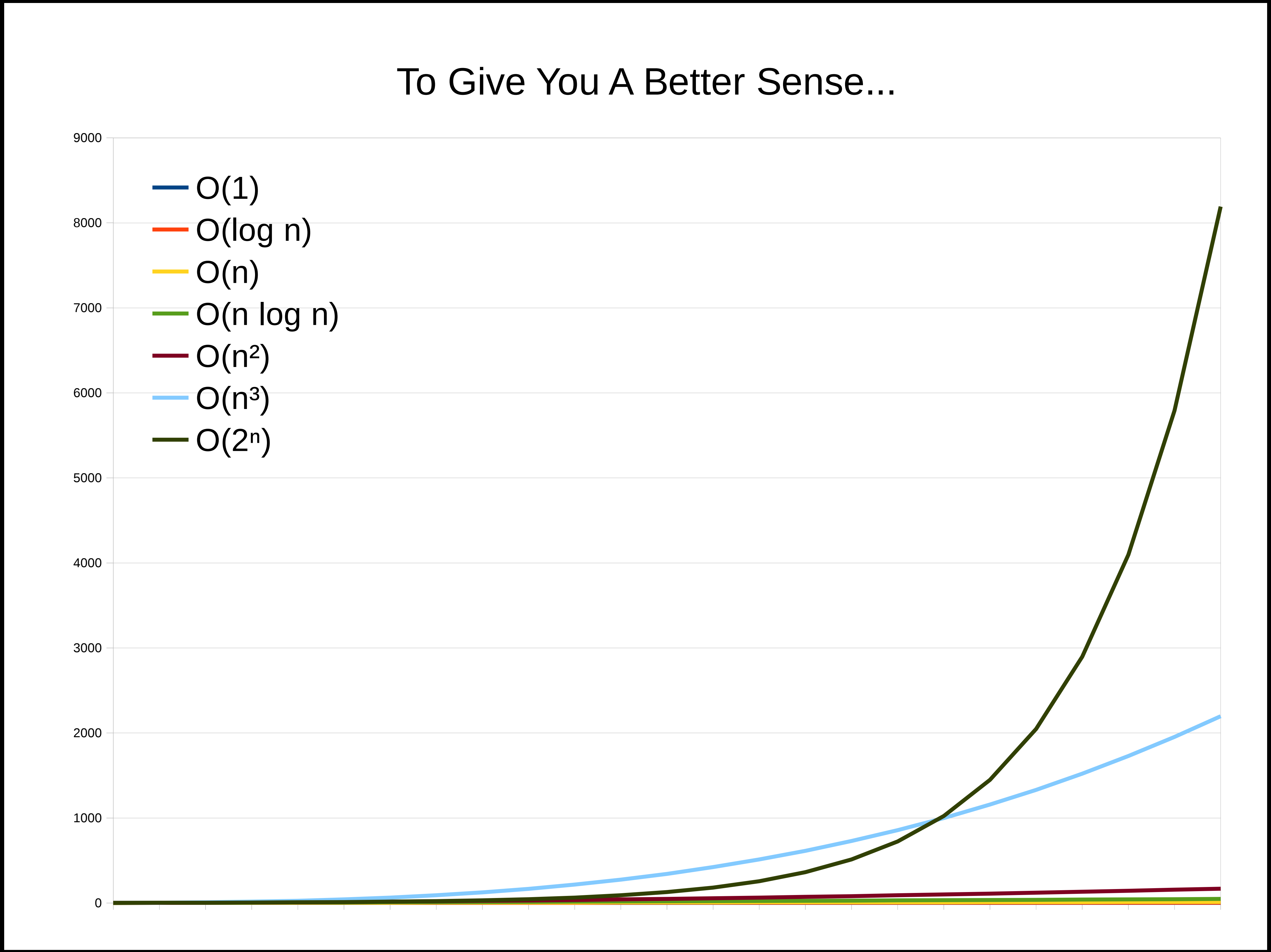


A visual comparison of growth rates


Growth Rates, Part One
16
14
12
10
8
6
4
2
0
O(1)
O(log n)
O(n)


Growth Rates, Part Two
250
200
150
100
50
0
O(n)
O(n log n)
O(n²)


Growth Rates, Part Three
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
O(n²)
O(n³)
O(2ⁿ)


To Give You A Better Sense...
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
O(1)
O(log n)
O(n)
O(n log n)
O(n²)
O(n³)
O(2ⁿ)

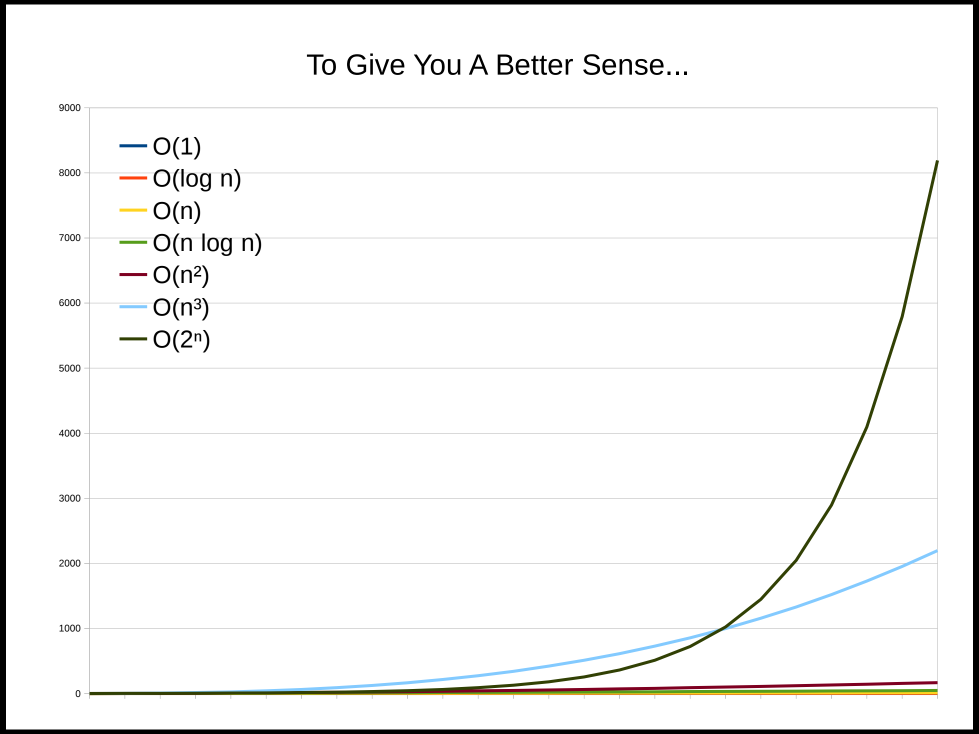
Tight is more meaningful

If $T(n)$ is $O(n)$

It is also true that $T(n)$ is $O(n^3)$

And it is also true that $T(n)$ is $O(2^n)$

But what does it mean???



The closest Big-O is the most descriptive of the overall worst-case behavior

Tightening the bounds

Big-O: upper bound

$T(n)$ is $O(f(n))$

if there exist constants **k** and $\mathbf{n_0}$ such that for all $n \geq n_0$ $T(n) \leq k\, f(n)$

Grows no faster than f(n)

Tightening the bounds

Big-O: upper bound

T(n) is O(f(n))

if there exist constants **k** and **n_0** such that for all $n \geq n_0$ $T(n) \leq k\, f(n)$

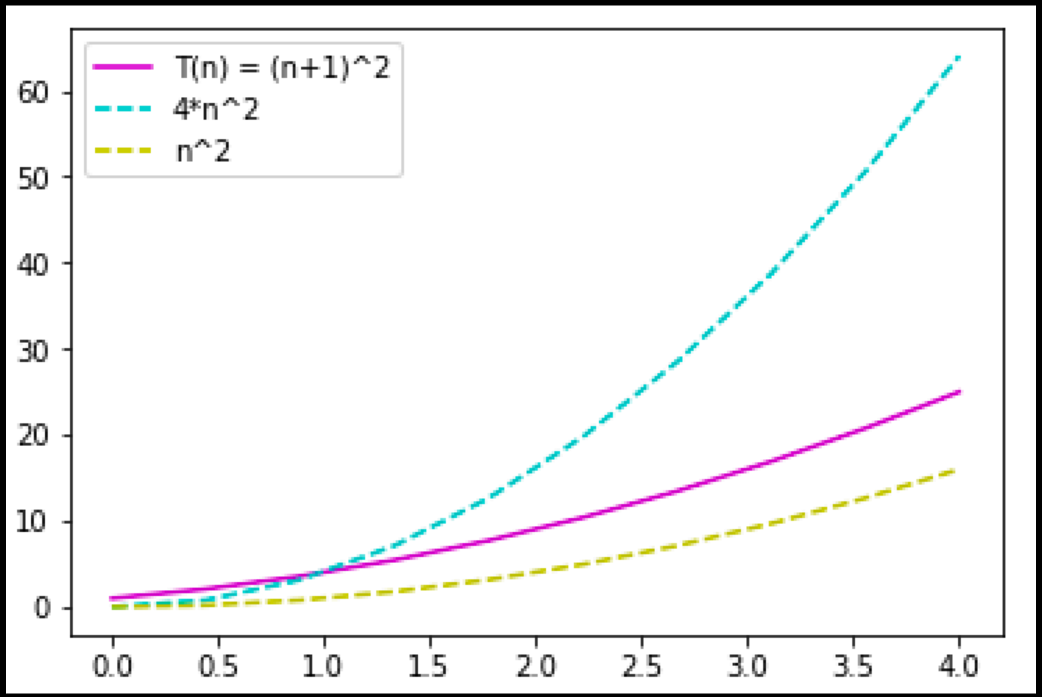
Grows no faster than f(n)

Omega: lower bound

T(n) is Ω(f(n))

if there exist constants **k** and **n_0** such that for all $n \geq n_0$ $T(n) \geq k\, f(n)$

Grows at least as fast as f(n)

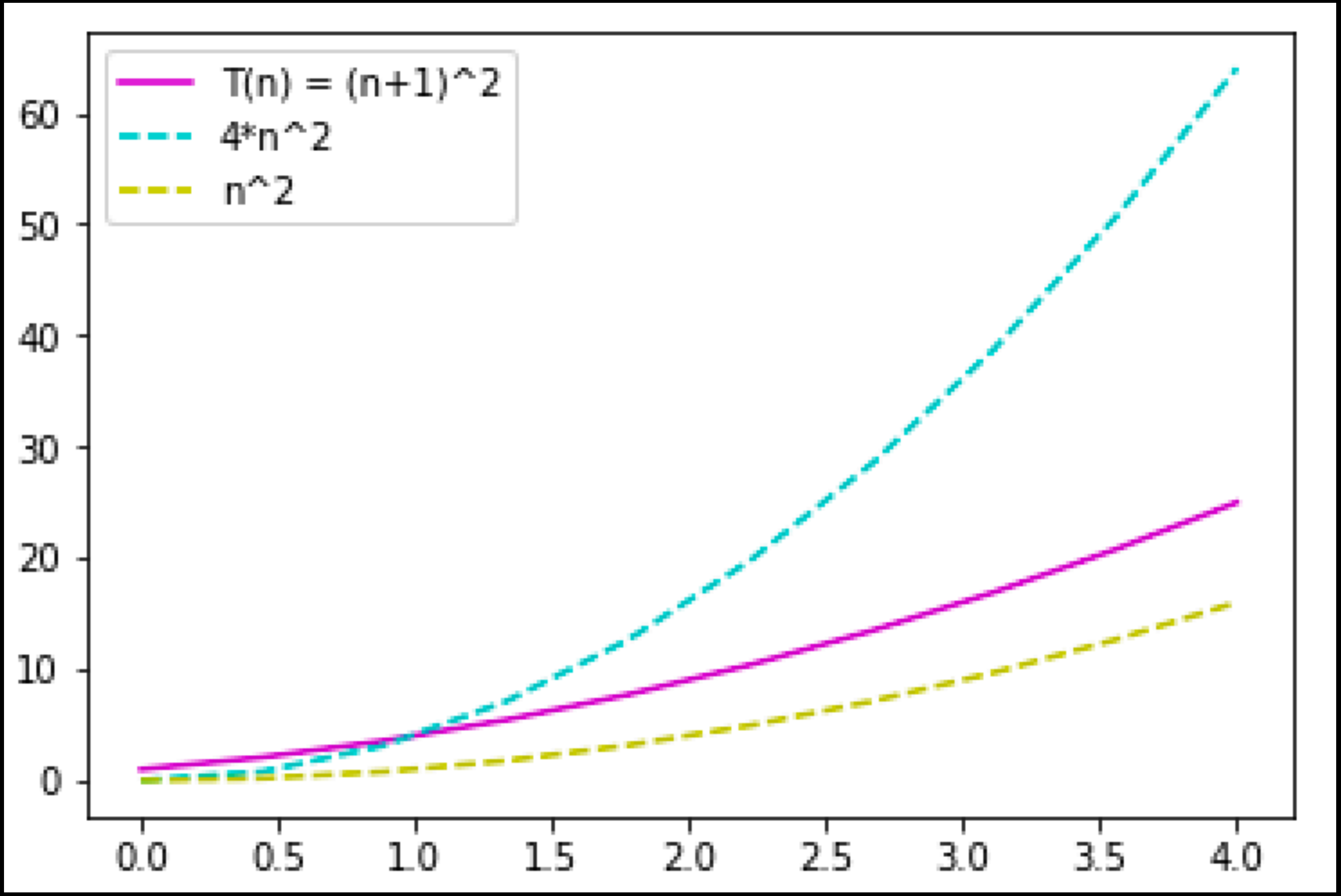


Tightening the bounds

Theta: tight bound

$T(n)$ is $\Theta(f(n))$

Grows at the same rate as $f(n)$: iff both $T(n)$ is $O(f(n))$ and $\Omega(f(n))$



A numerical comparison of growth rates

n f(n)	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

What does Big-O describe?

"Long term" behavior of a function

Compare behavior of 2 algorithms

If algorithm A has runtime O(n) and algorithm B has runtime $O(n^2)$, **for large inputs** A will always be faster.

If algorithm A has runtime O(n), doubling the size of the input will double the runtime

Analyze algorithm behavior with growing input

What can't Big-O describe?

The actual runtime of an algorithm

$10^{100}n = O(n)$

$10^{-100}n = O(n)$

How an algorithm behaves on small input

$n^3 = O(n^3)$

$10^6 = O(1)$

To summarize Big-O

It is a means of describing the growth rate of a function

It ignores all but the dominant term

It ignores constants

Allows for quantitative ranking of algorithms

Allows for quantitative reasoning about algorithms