Sorting

Tiziana Ligorio

Today’s Plan

Sorting algorithms and their analysis

Sorting

Rearranging a sequence into increasing (decreasing) order!

Several approaches

Can do it in may ways

What is the best way?

Let’s find out using Big-O

Lecture Activity

Write **pseudocode** to sort an array.

543	3	523	76	200	158	195	108	43	274	100	14	599

There are many approaches to sorting
We will look at some comparison-based approaches here

Selection Sort

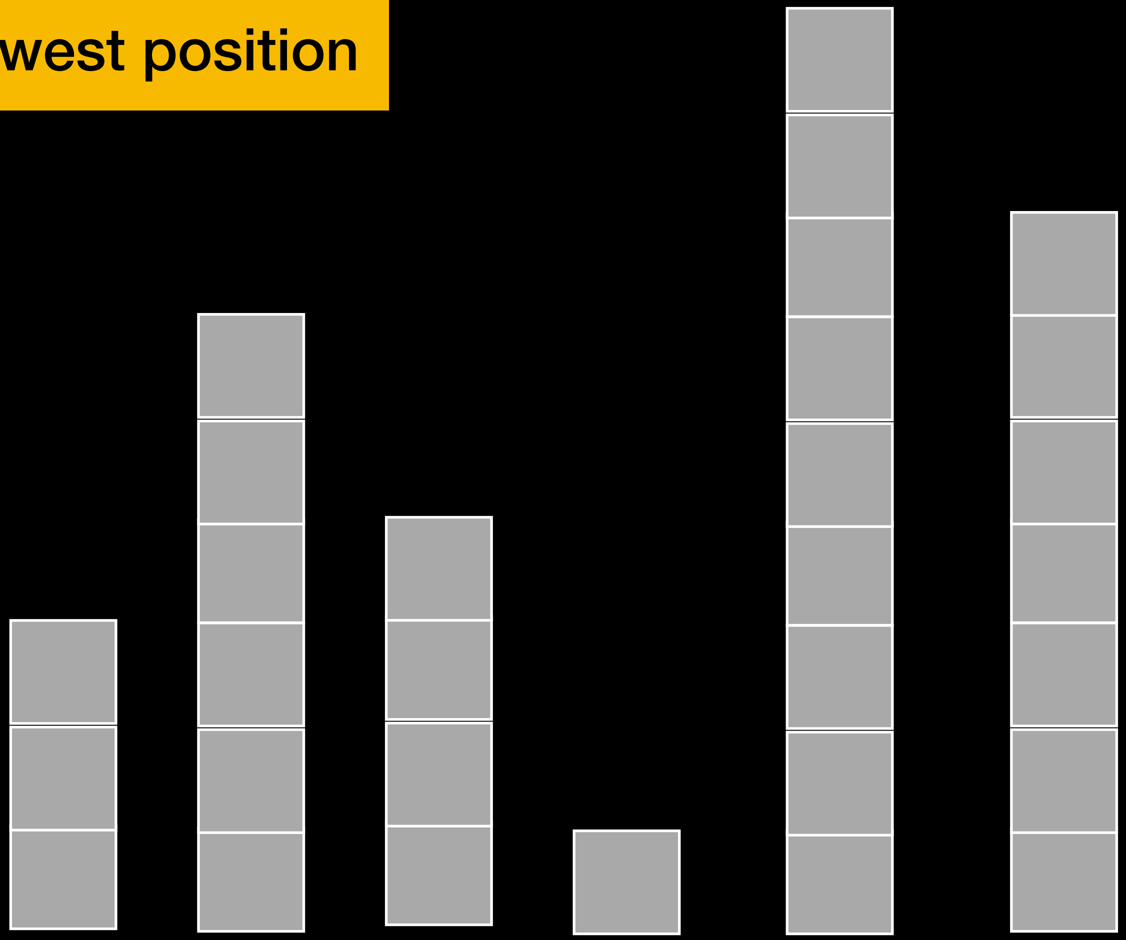
Selection Sort

⬜ Unsorted

🟦 Sorted

Find smallest element and move it at lowest position

1st Pass



Selection Sort

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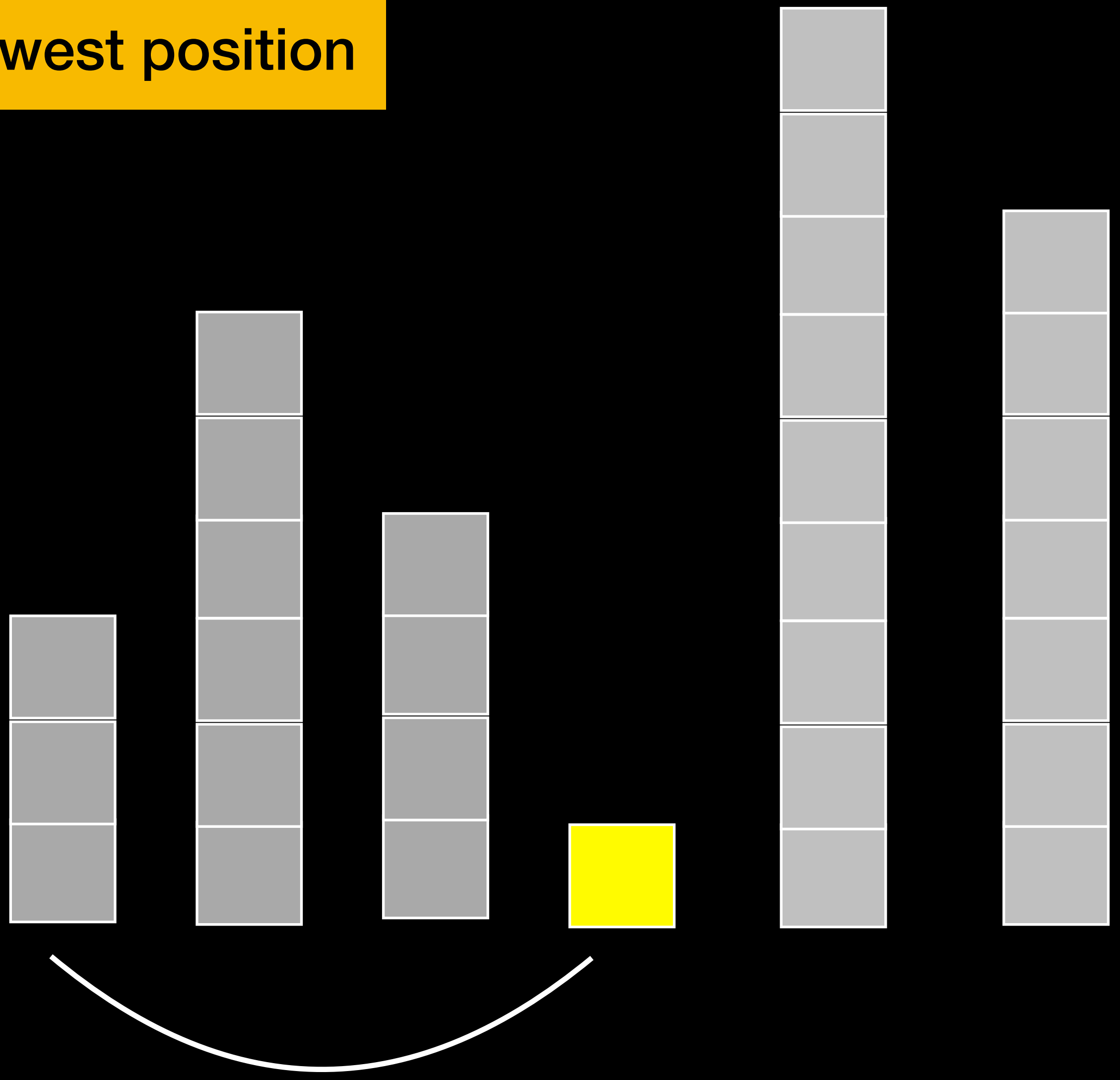

Swap

Selection Sort

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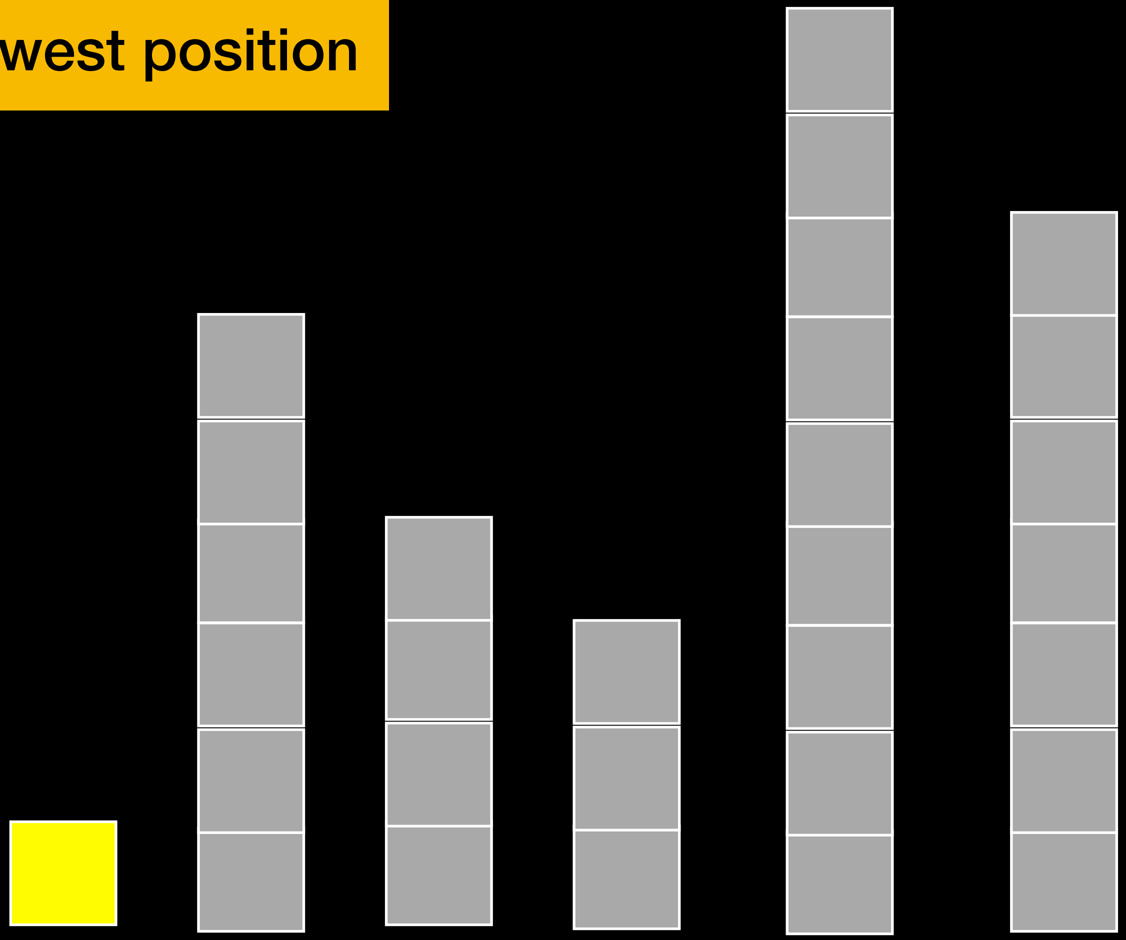
1st Pass



Selection Sort

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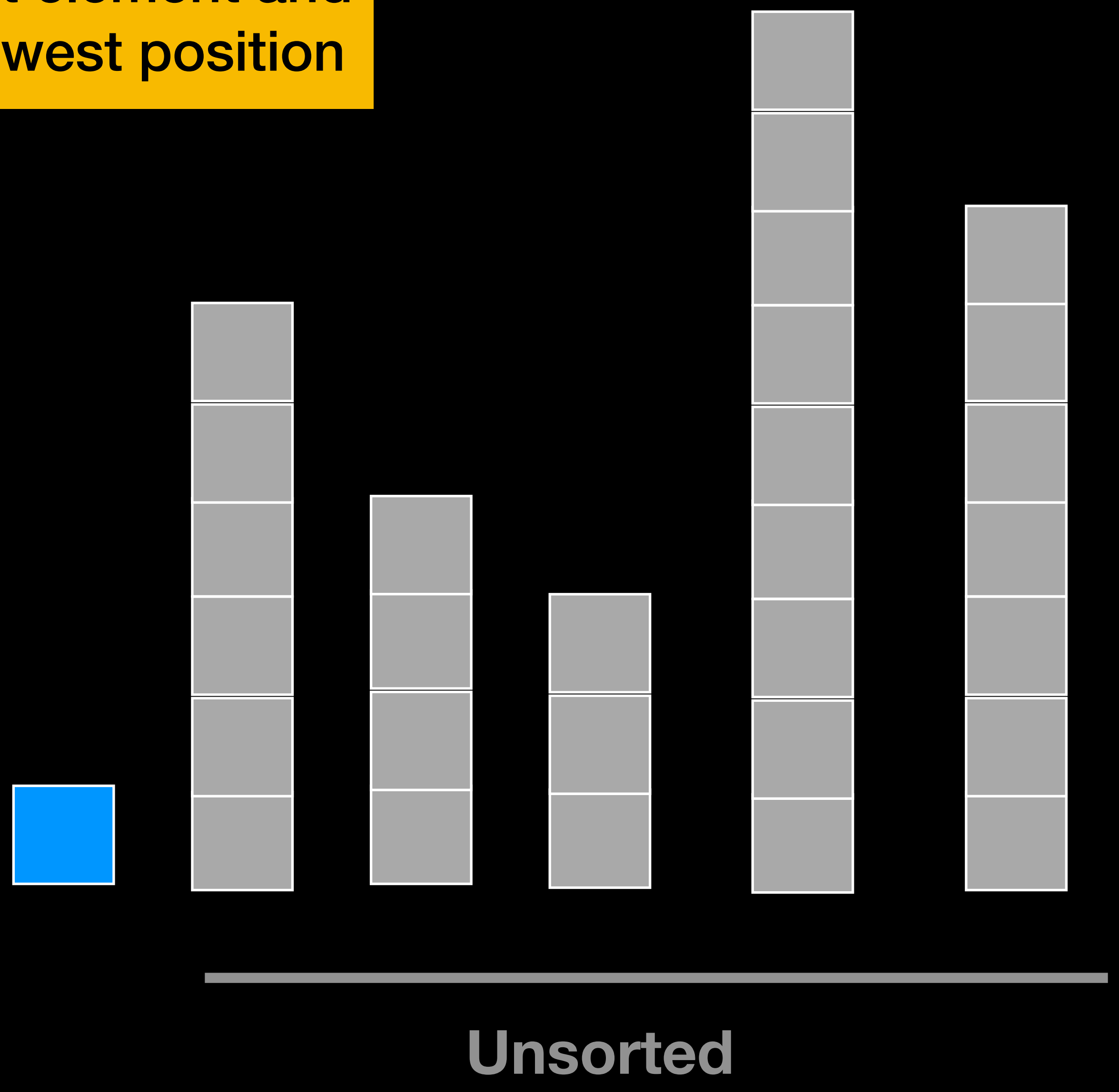
Find smallest element and move it at lowest position

2nd Pass



Selection Sort

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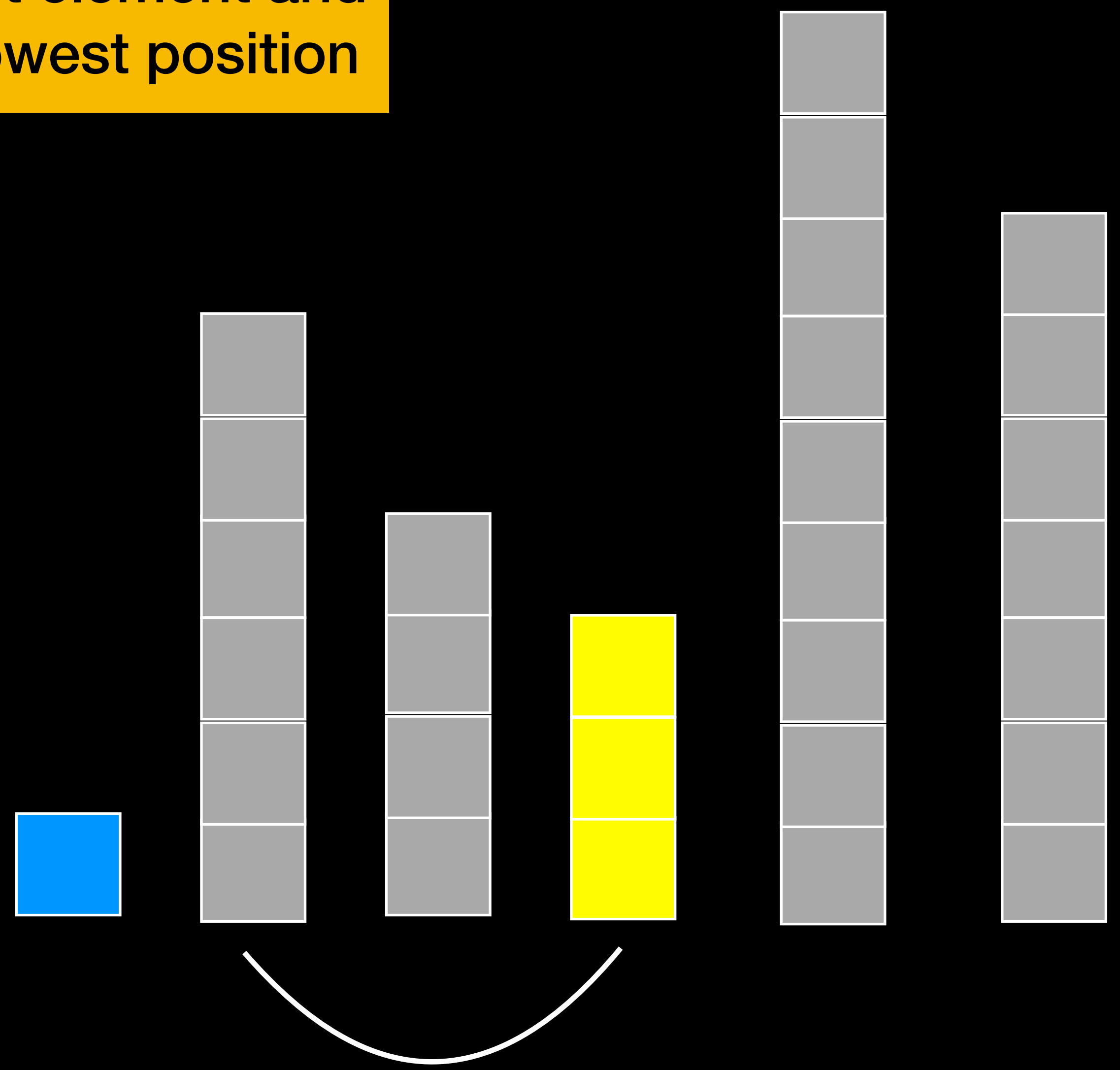

Swap

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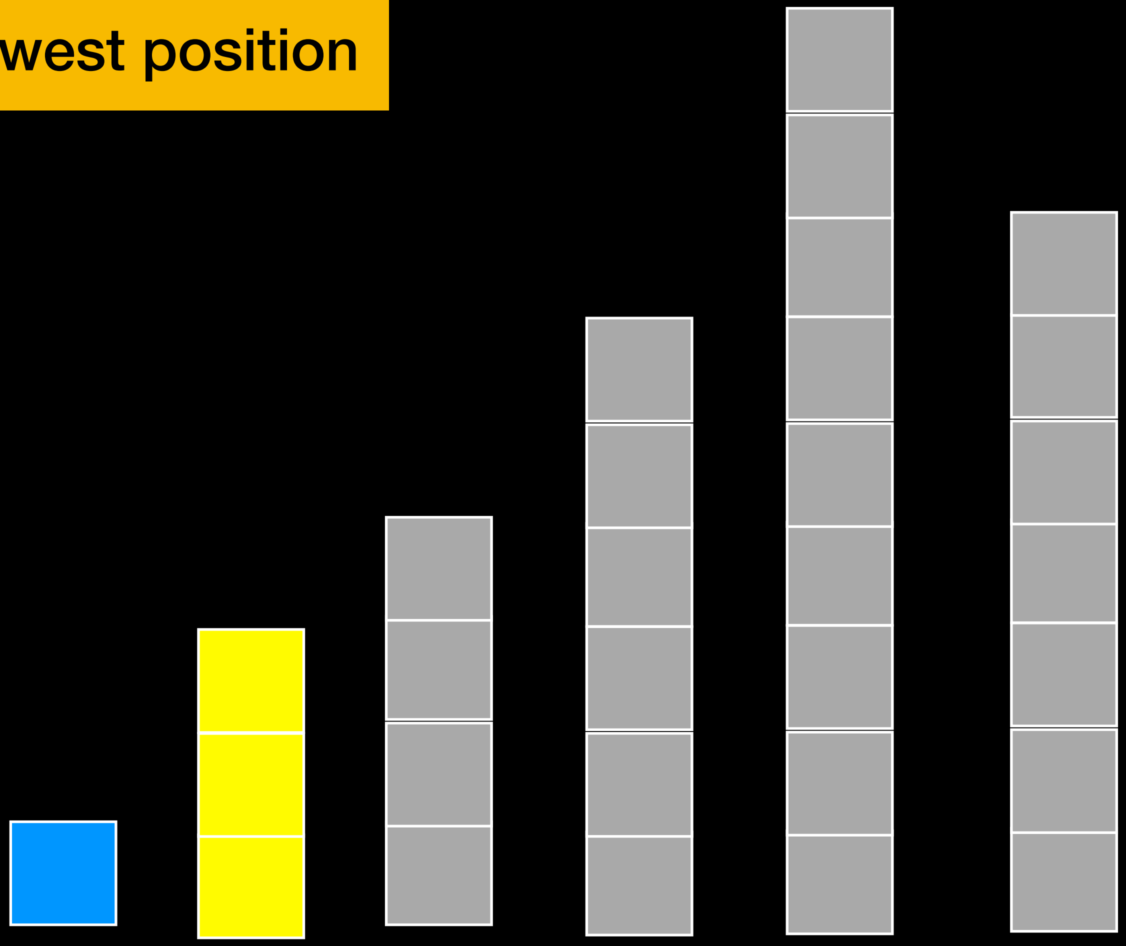
2nd Pass



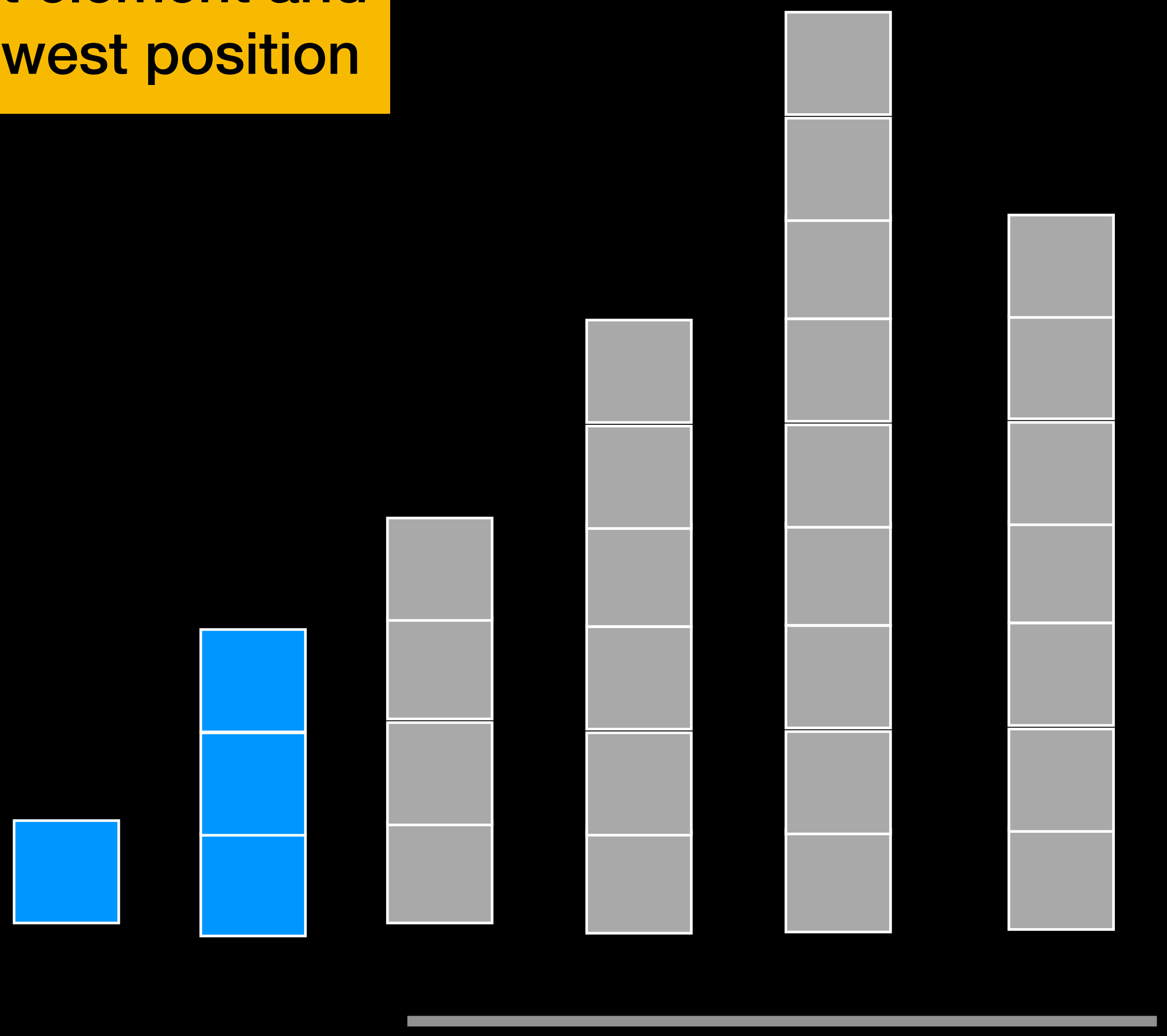
Selection Sort

▨ Unsorted

🟦 Sorted

Find smallest element and move it at lowest position

3rd Pass



Selection Sort

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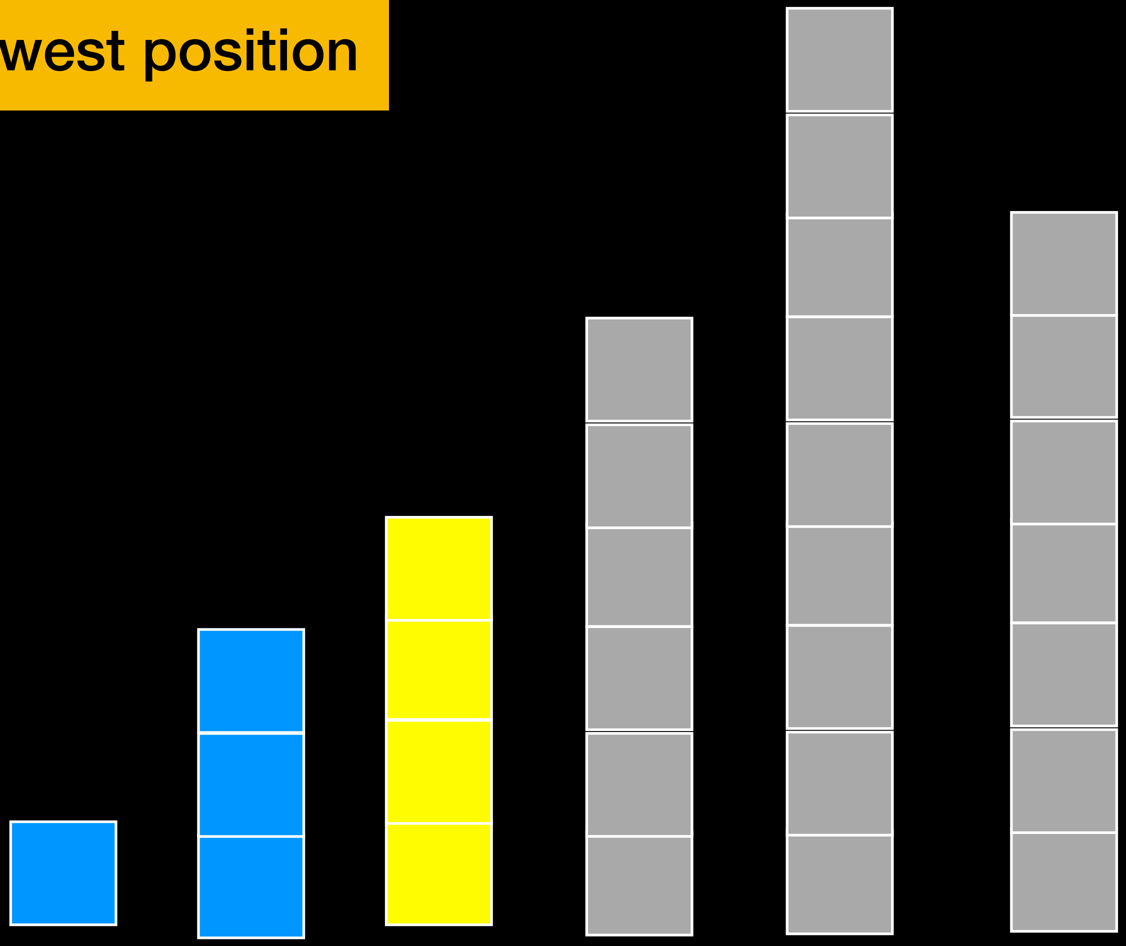
3rd Pass



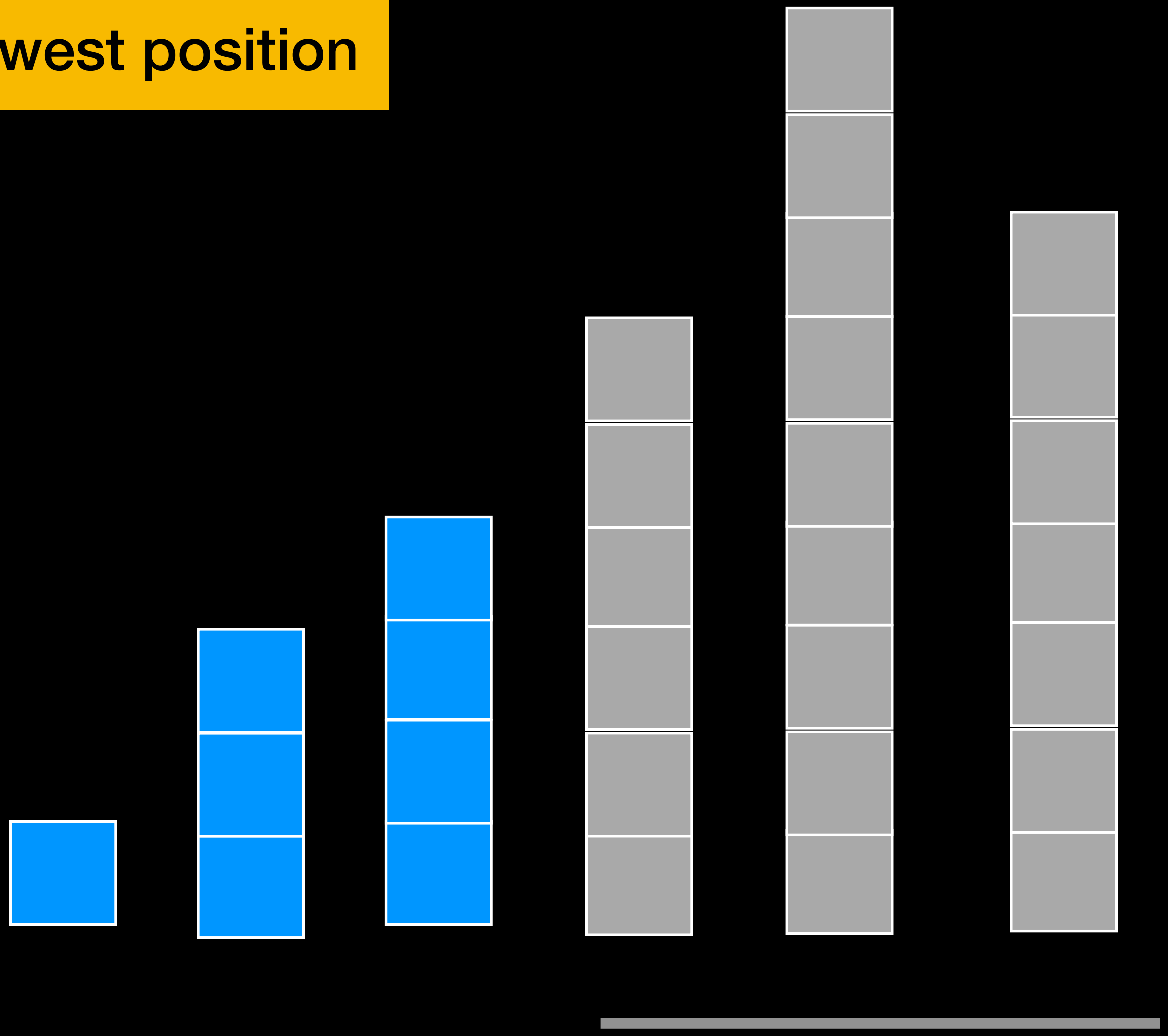
Selection Sort

⬜ Unsorted

🟦 Sorted

Find smallest element and move it at lowest position

4th Pass



Selection Sort

▨ **Unsorted**

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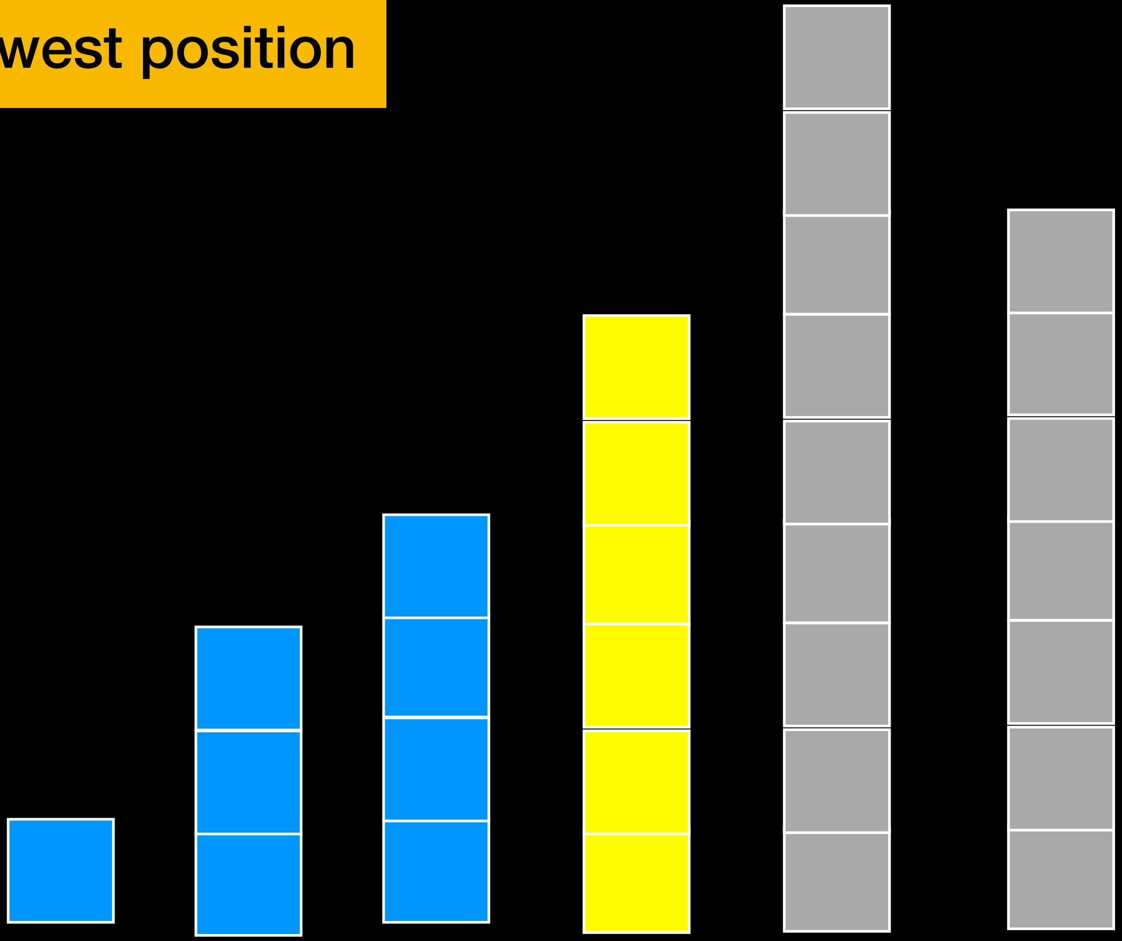
Find smallest element and move it at lowest position

4th Pass



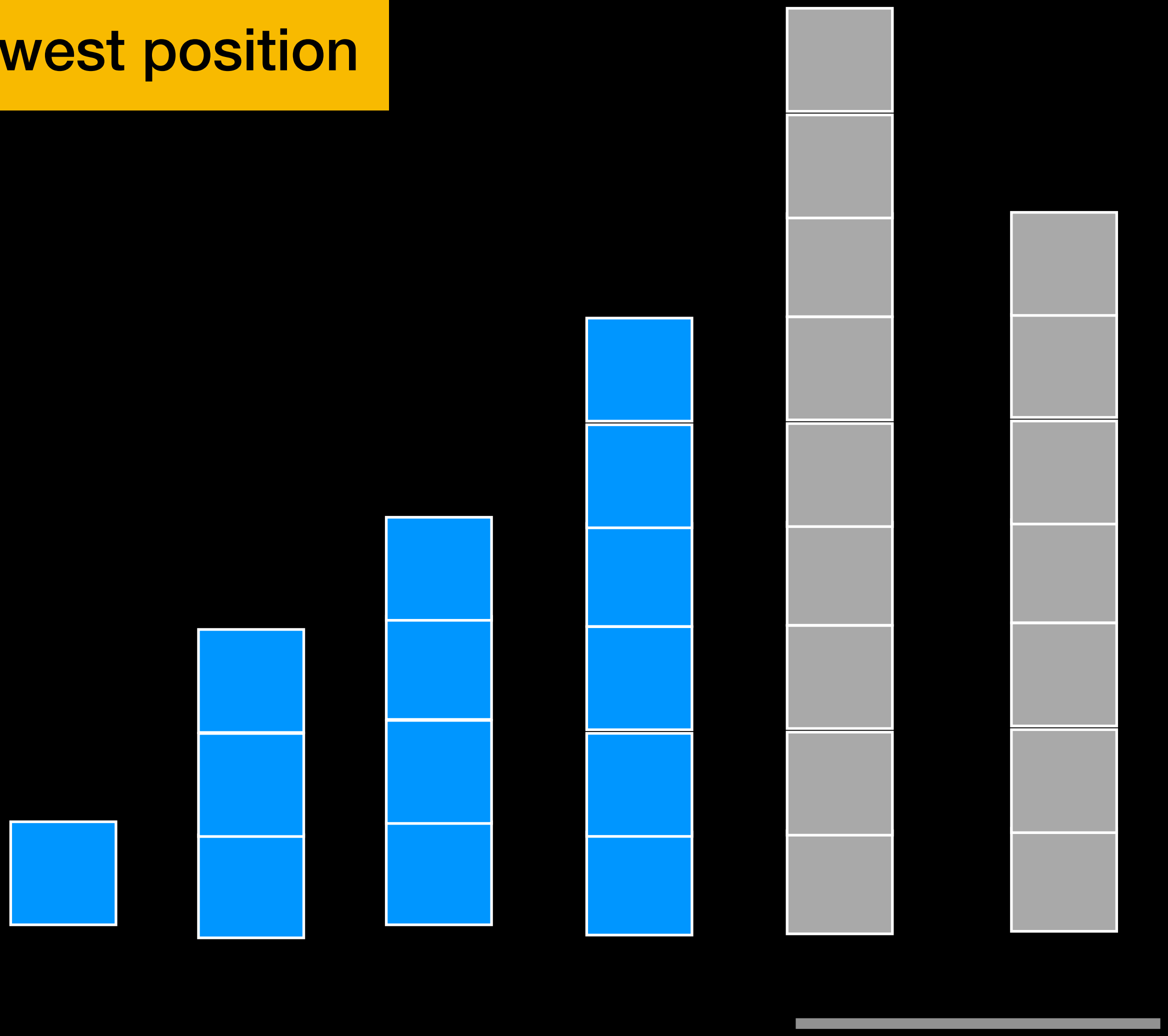
Selection Sort

⬜ **Unsorted**

🟦 **Sorted**

Find smallest element and move it at lowest position

5th Pass



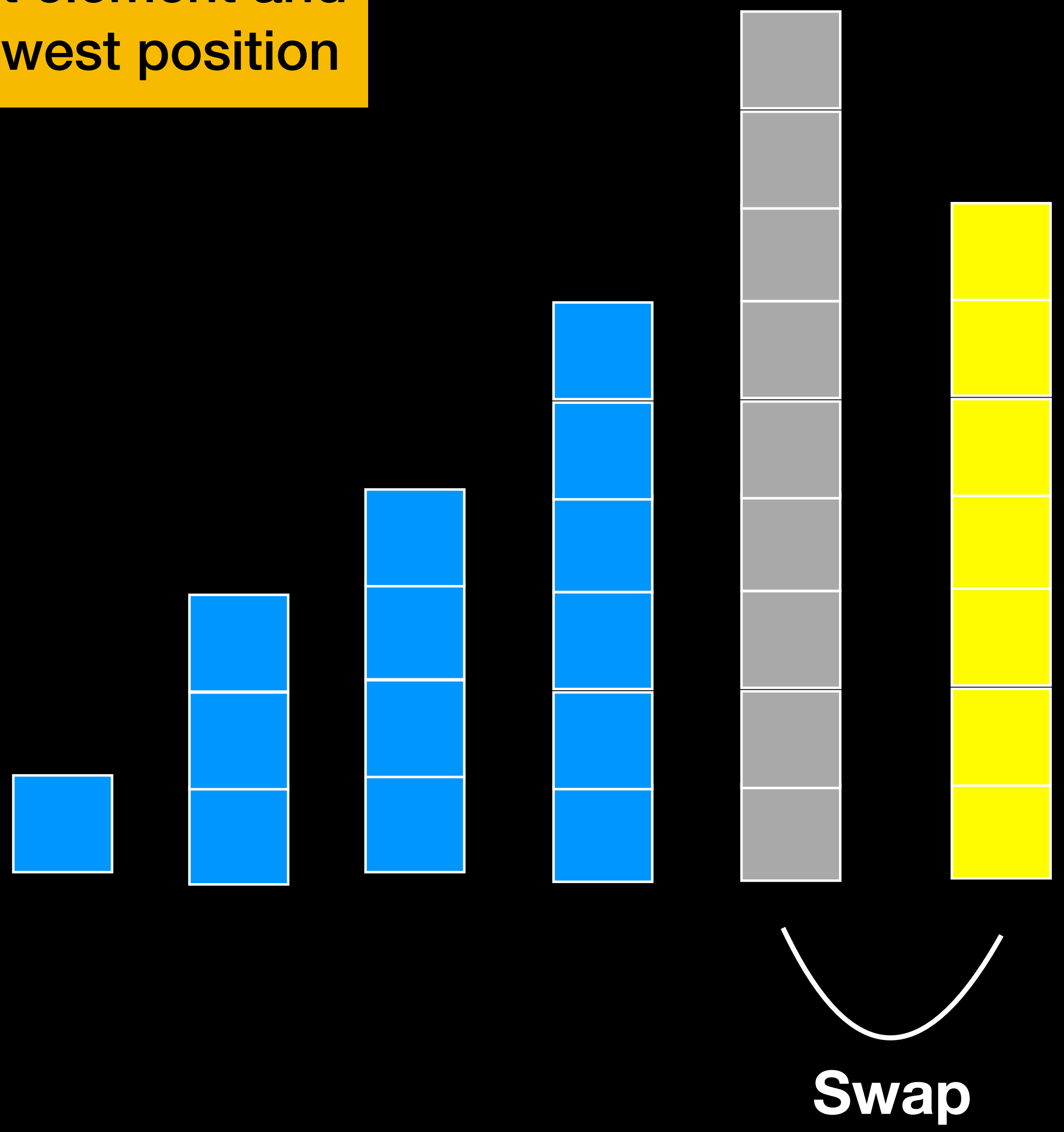
Selection Sort

▨ Unsorted

🟦 Sorted

Find smallest element and move it at lowest position

5th Pass



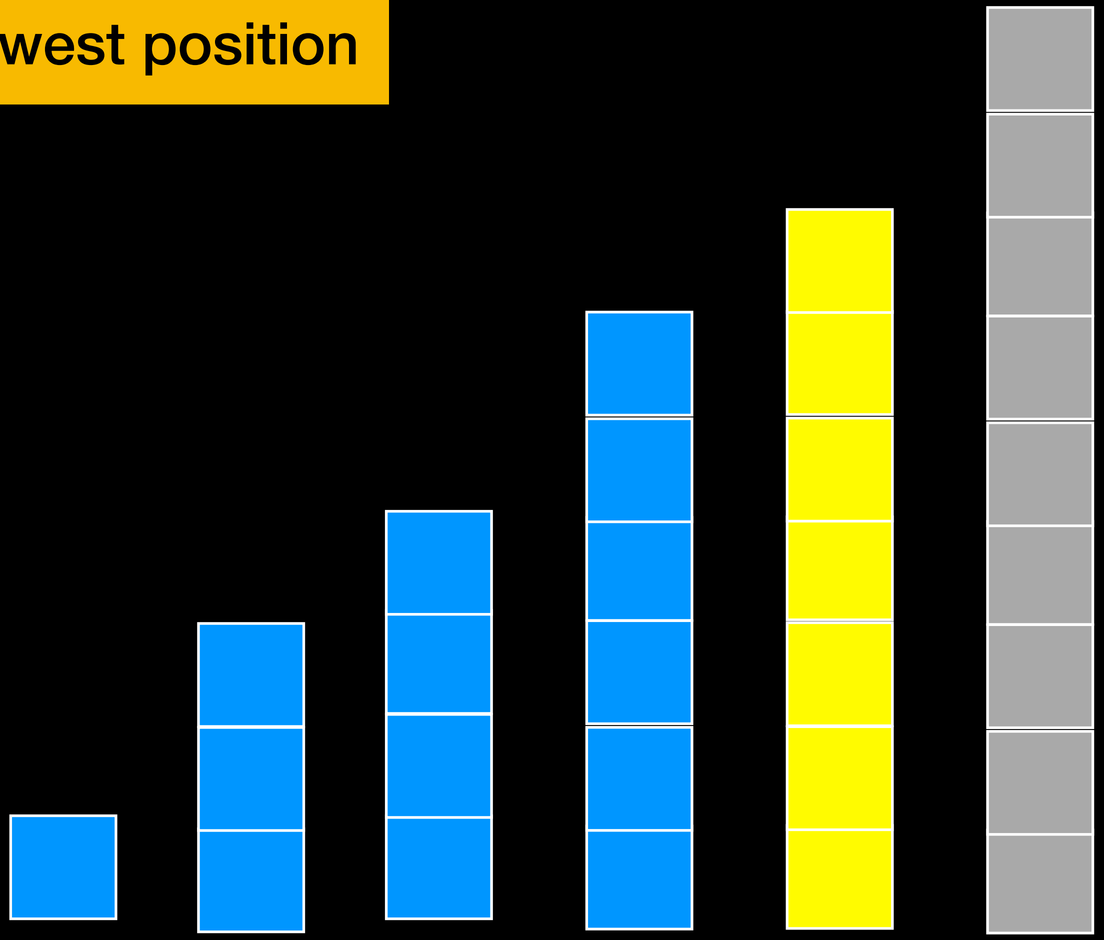
Selection Sort

⬜ **Unsorted**

🟦 **Sorted**

Find smallest element and move it at lowest position

5th Pass



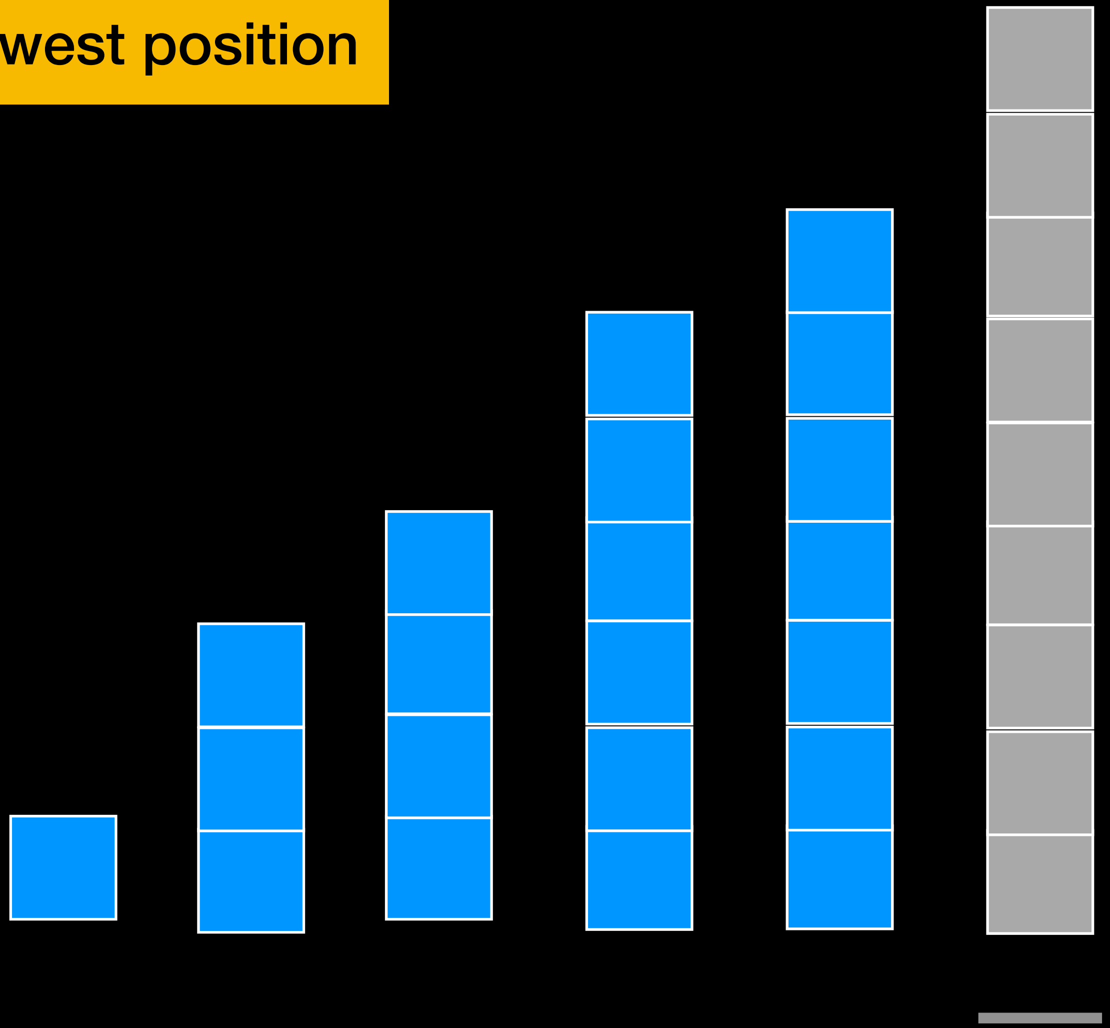
Selection Sort

▨ **Unsorted**

🟦 **Sorted**

Find smallest element and move it at lowest position

6th Pass

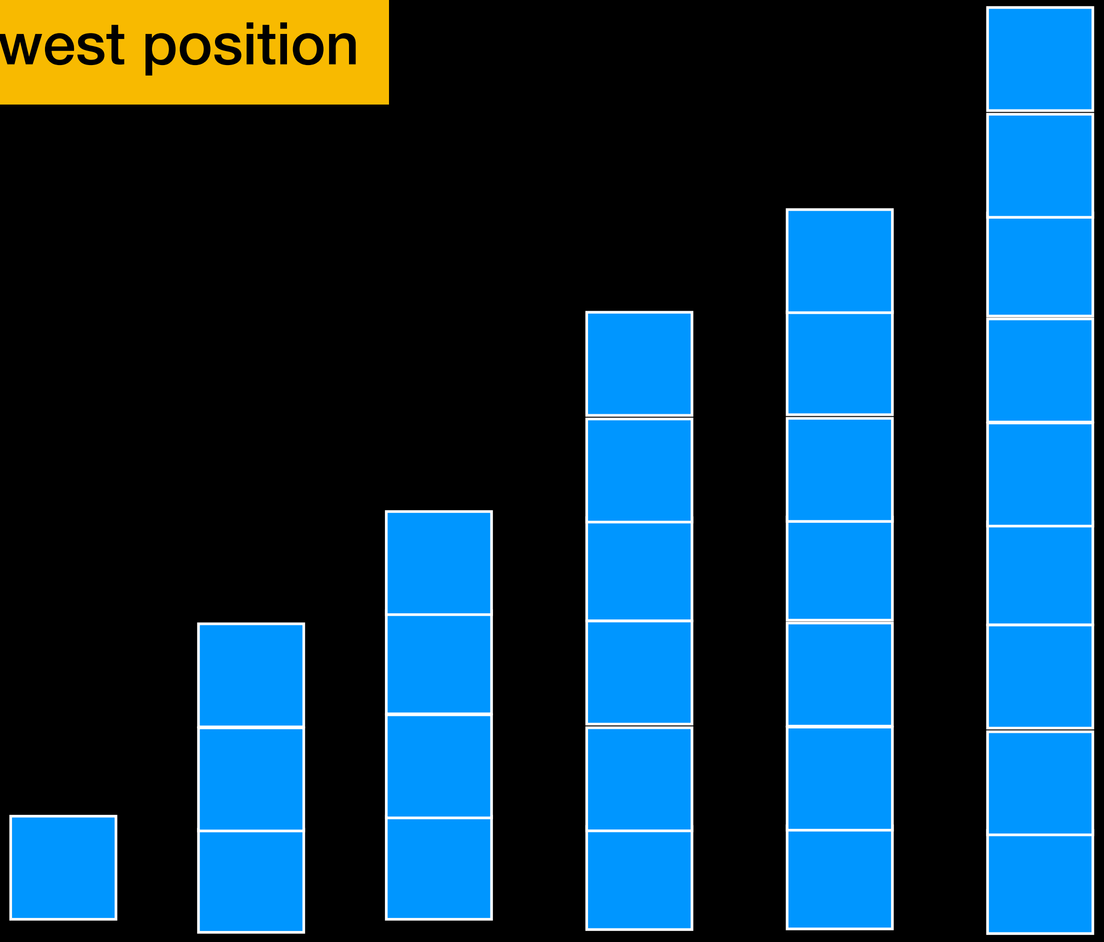


Selection Sort

▨ **Unsorted**

🟦 **Sorted**

Find smallest element and move it at lowest position



Selection Sort

Find the smallest item and move it at position 1

Find the next-smallest item and move it at position 2

. . .

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Find second smallest: look at **n-1** elements

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Find second smallest: look at **n-1** elements

Find third smallest: look at **n-2** elements

. . .

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

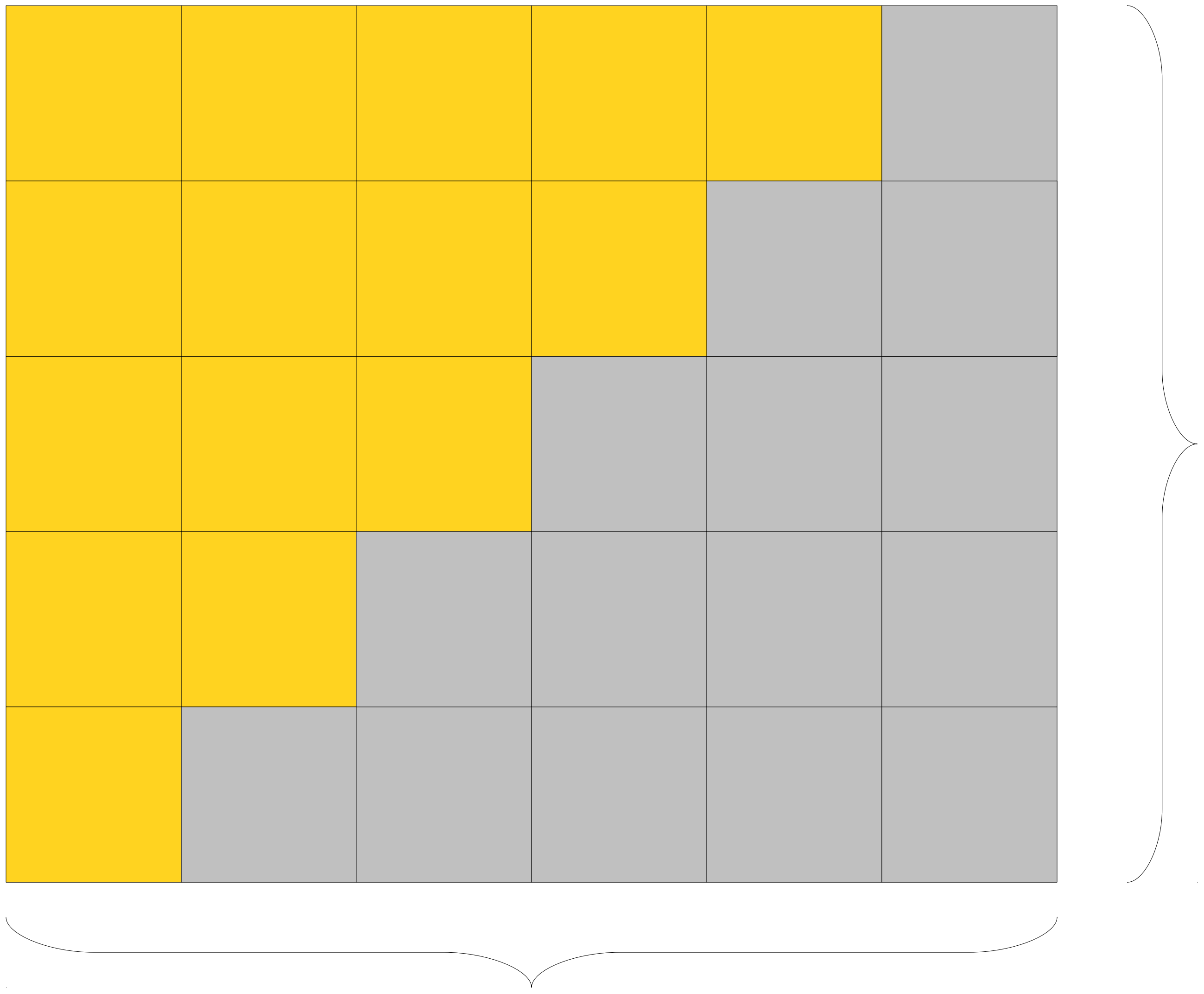
Find second smallest: look at **n-1** elements

Find third smallest: look at **n-2** elements

. . .

Total work: **n + (n-1) + (n-2) + . . . +1**

$$n + (n\text{-}1) + \ldots + 2 + 1 = n(n+1) / 2$$



Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons $+ n$ data moves $= O(\ )$?

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Selection Sort Analysis

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$T(n) = (n^2+n) / 2 + n = O(\ )$?

Ignore constant

Ignore non-dominant terms

Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons $+ n$ data moves $= O(\ )$?

$T(n) = (n^2+n) / 2 + n = O(\mathbf{n^2})$

Ignore constant

Ignore non-dominant terms

Selection Sort Analysis

$T(n)$ = $n(n+1) / 2$ comparisons + n data moves = $O(\ )$?

$T(n) = (n^2+n) / 2 + n = O(n^2)$

Selection Sort run time is $\mathbf{O(n^2)}$

```
template<typename ItemType>
void selectionSort(ItemType the_array[], size_t size)
{
   // first = index of the first item in the subarray of items yet
   //        to be sorted;
   // smallest = index of the smallest item found
   for (int first = 0; first < size; first++)
   {
      // At this point, the_array[0 ...first-1] is sorted, and its
      // entries are <= those in the_array[first ... size-1].
      // Select the smallest entry in the_array[first ... size-1]
      int smallest_index = findIndexOfSmallest(the_array, first,
                                                             size);
      // Swap the smallest entry, the_array[smallest_index], with
      // the first in the unsorted subarray the_array[first]
      swap(the_array[smallest_index], the_array[first]);
   }  // end for
}  // end selectionSort
```

```
template<typename ItemType>
void selectionSort(ItemType the_array[], size_t size)
{
   // first = index of the first item in the subarray of items yet
   //        to be sorted;
   // smallest = index of the smallest item found
   for (int first = 0; first < size; first++)
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      int smallest_index = findIndexOfSmallest(the_array, first,
                                                             size);
      // Swap the smallest entry, the_array[smallest_index], with
      // the first in the unsorted subarray the_array[first]
      swap(the_array[smallest_index], the_array[first]);
   }  // end for
}  // end selectionSort
```

Pass

O(n)

O(n)

O(n^2)

Stability

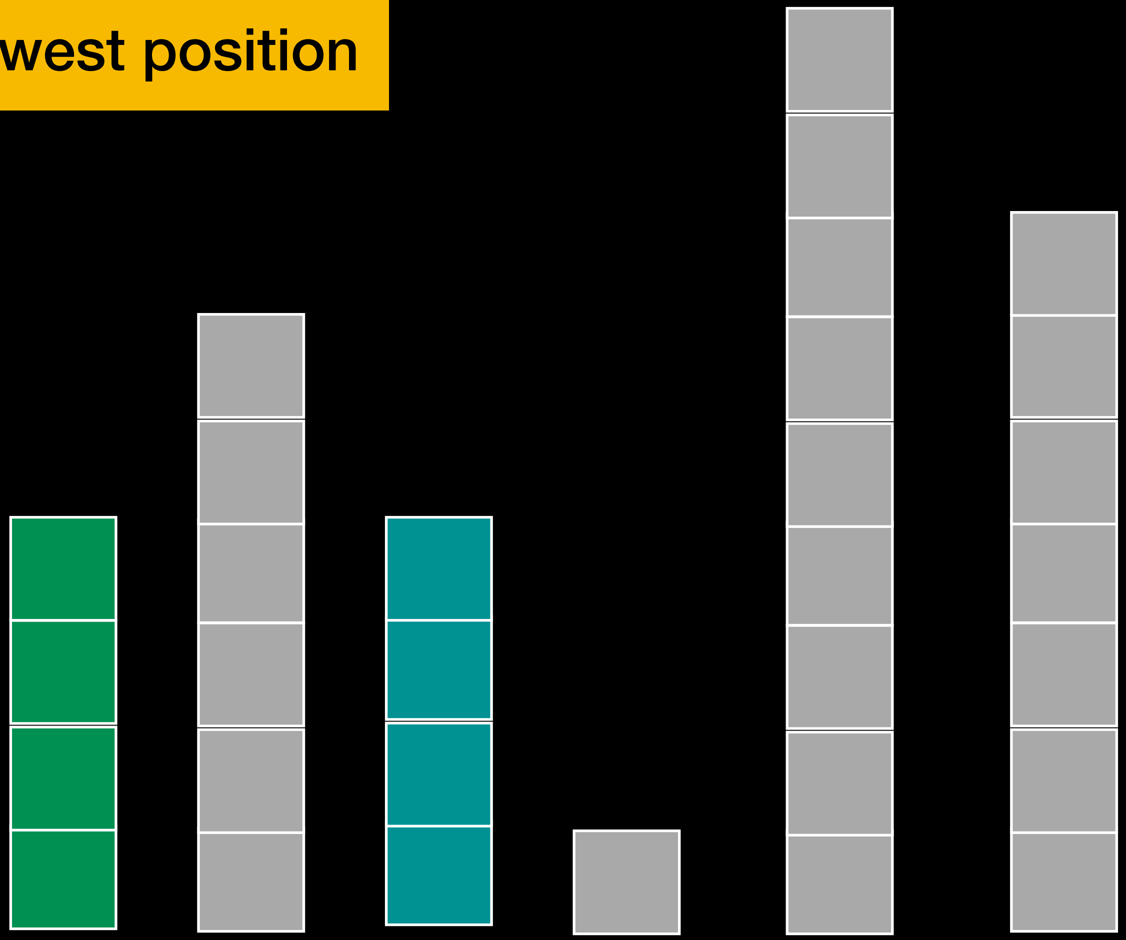
A sorting algorithm is Stable if elements that are equal remain is same order relative to each other after sorting

Selection Sort

⬜ **Unsorted**

🟦 **Sorted**

Find smallest element and move it at lowest position



Selection Sort

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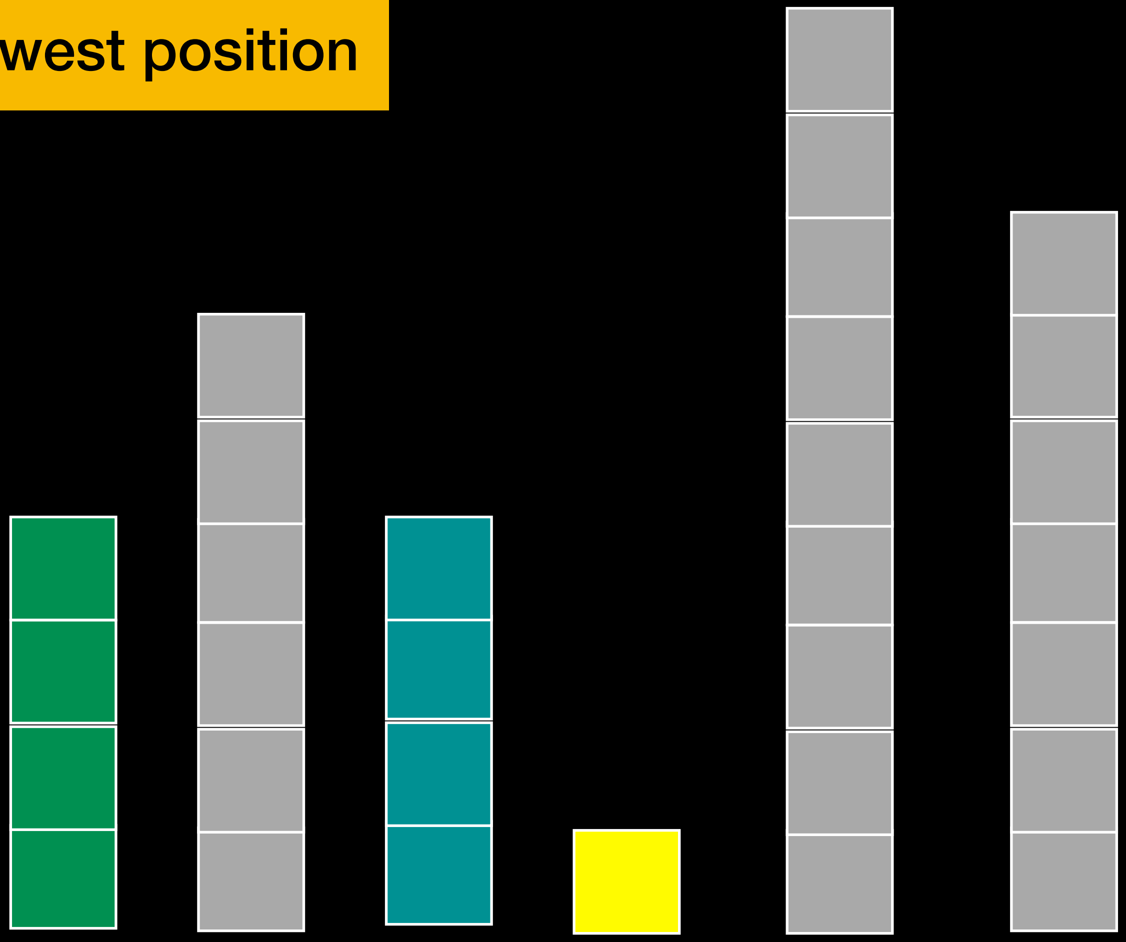

Selection Sort

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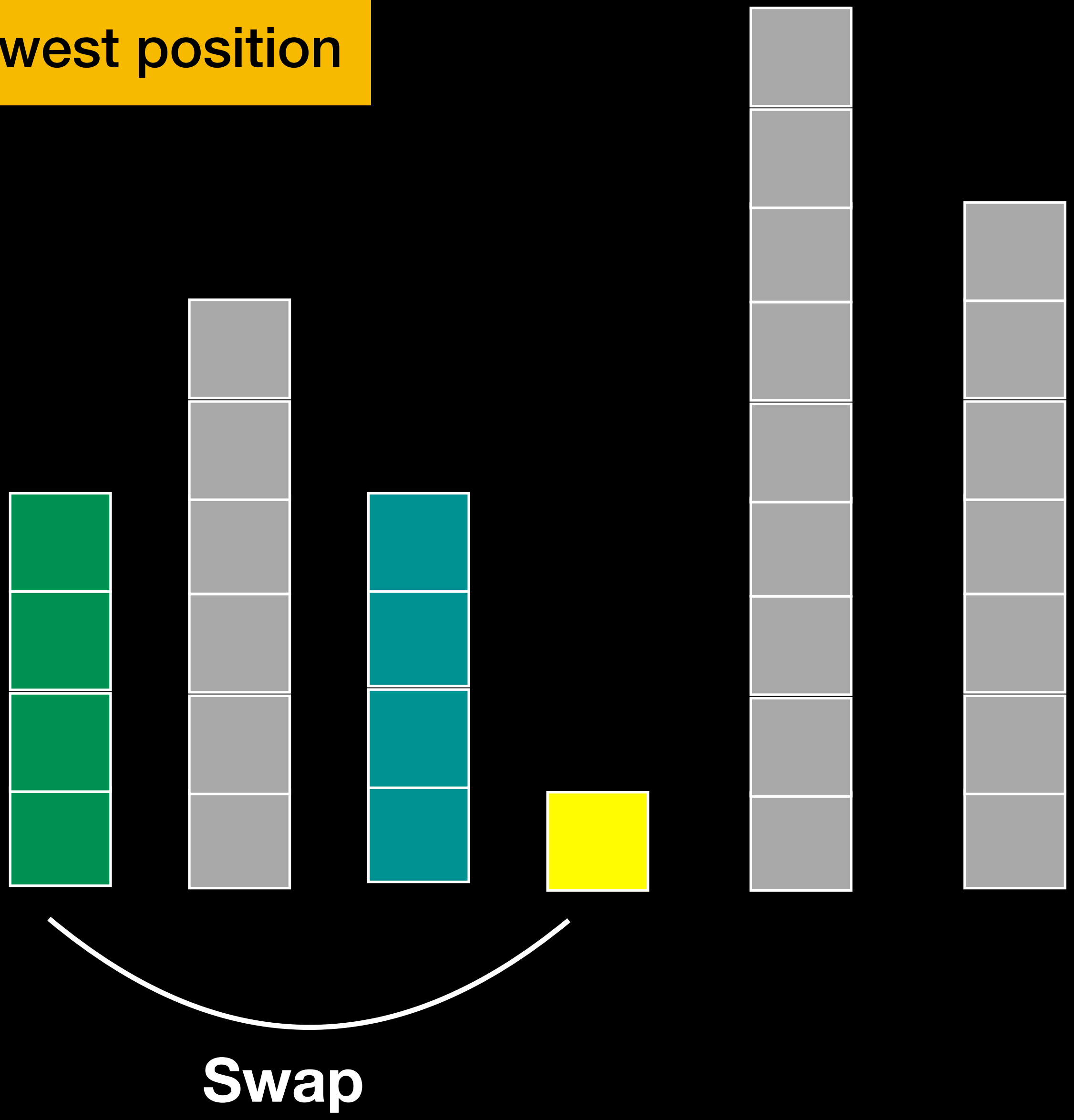
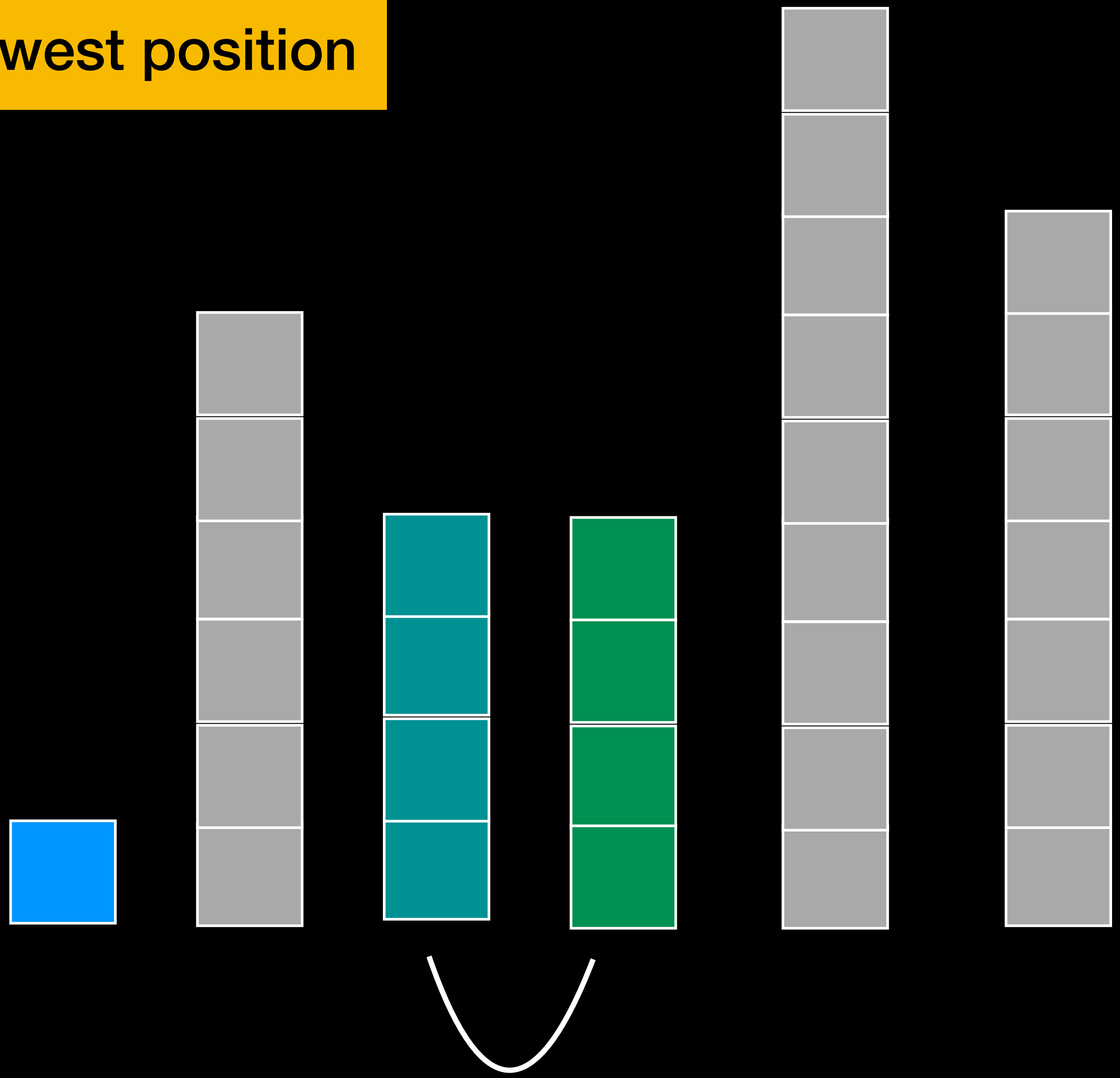


Selection Sort

▨ Unsorted

🟦 Sorted

Find smallest element and move it at lowest position



Unstable

Selection Sort Analysis

Execution time DOES NOT depend on initial arrangement of data => ALWAYS $O(n^2)$

$O(n^2)$ comparisons

Good choice for small **n** and/or data moves are costly ($O(n)$ data moves)

Unstable

Understanding O(n^2)

100	14	3	43	200	274

T(n)

Understanding $O(n^2)$

100	14	3	43	200	274

$T(n)$

100	14	3	43	200	274	523	108	76	195	599	158

$$T(2n) \approx 4T(n)$$

$$(2n)^2 = 4n^2$$

Understanding $O(n^2)$

100	14	3	43	200	274

$T(n)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(3n) \approx 9T(n)$

$(3n)^2 = 9n^2$

Understanding $O(n^2)$ on large input

If size of input increases by factor of 100

Execution time increases by factor of 10,000

$T(100n) = 10{,}000T(n)$

Understanding O(n^2) on large input

If size of input increases by factor of 100

Execution time increases by factor of 10,000

$T(100n) = 10{,}000T(n)$

Assume $n = 100{,}000$ and $T(n) = 17$ seconds

Sorting 10,000,000 takes 10,000 longer

Understanding $O(n^2)$ on large input

If size of input increases by factor of 100

Execution time increases by factor of 10,000

$T(100n) = 10{,}000T(n)$

Assume $n = 100{,}000$ and $T(n) = 17$ seconds

Sorting 10,000,000 takes 10,000 longer

Sorting 10,000,000 entries takes ≈ 2 days

Multiplying input by 100 to go from 17sec to 2 days!!!

Raise your hand if you had Selection Sort

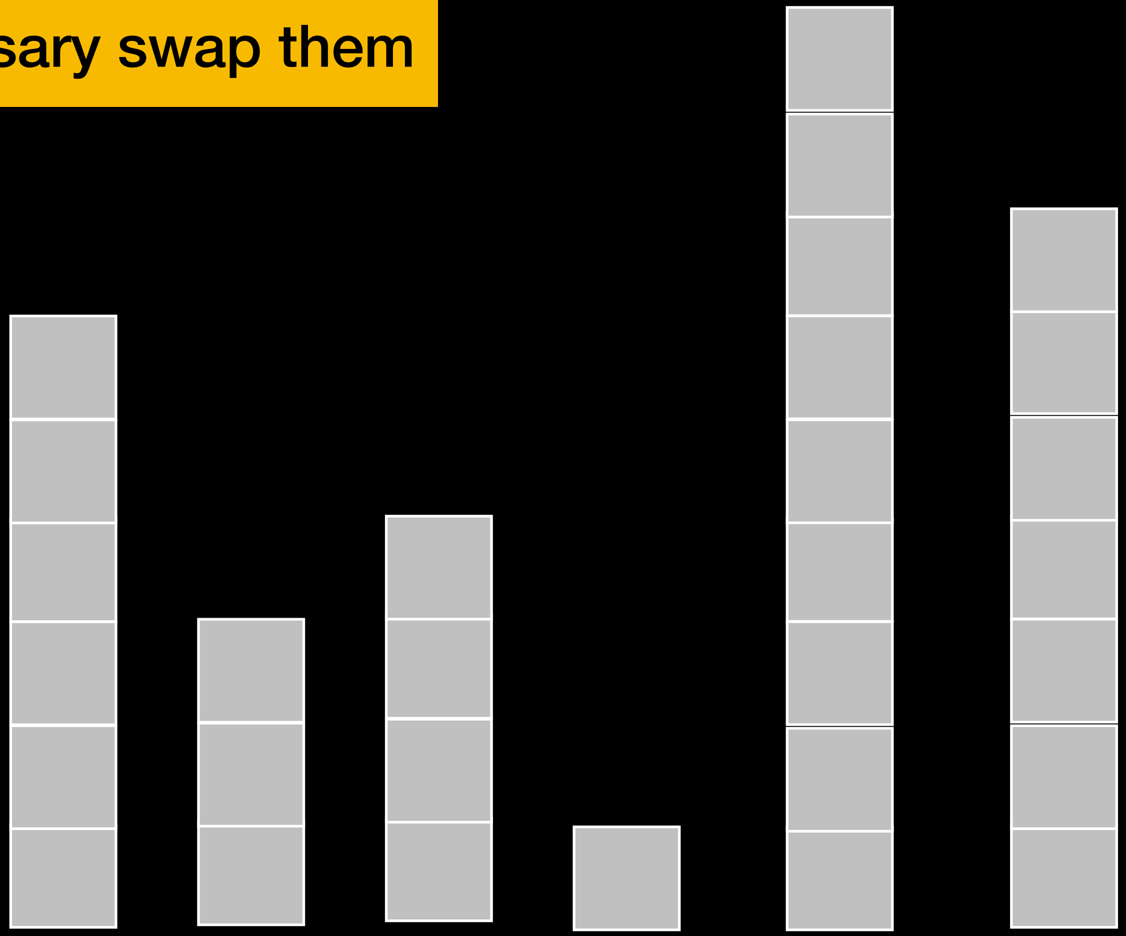
Bubble Sort

Bubble Sort

▨ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them



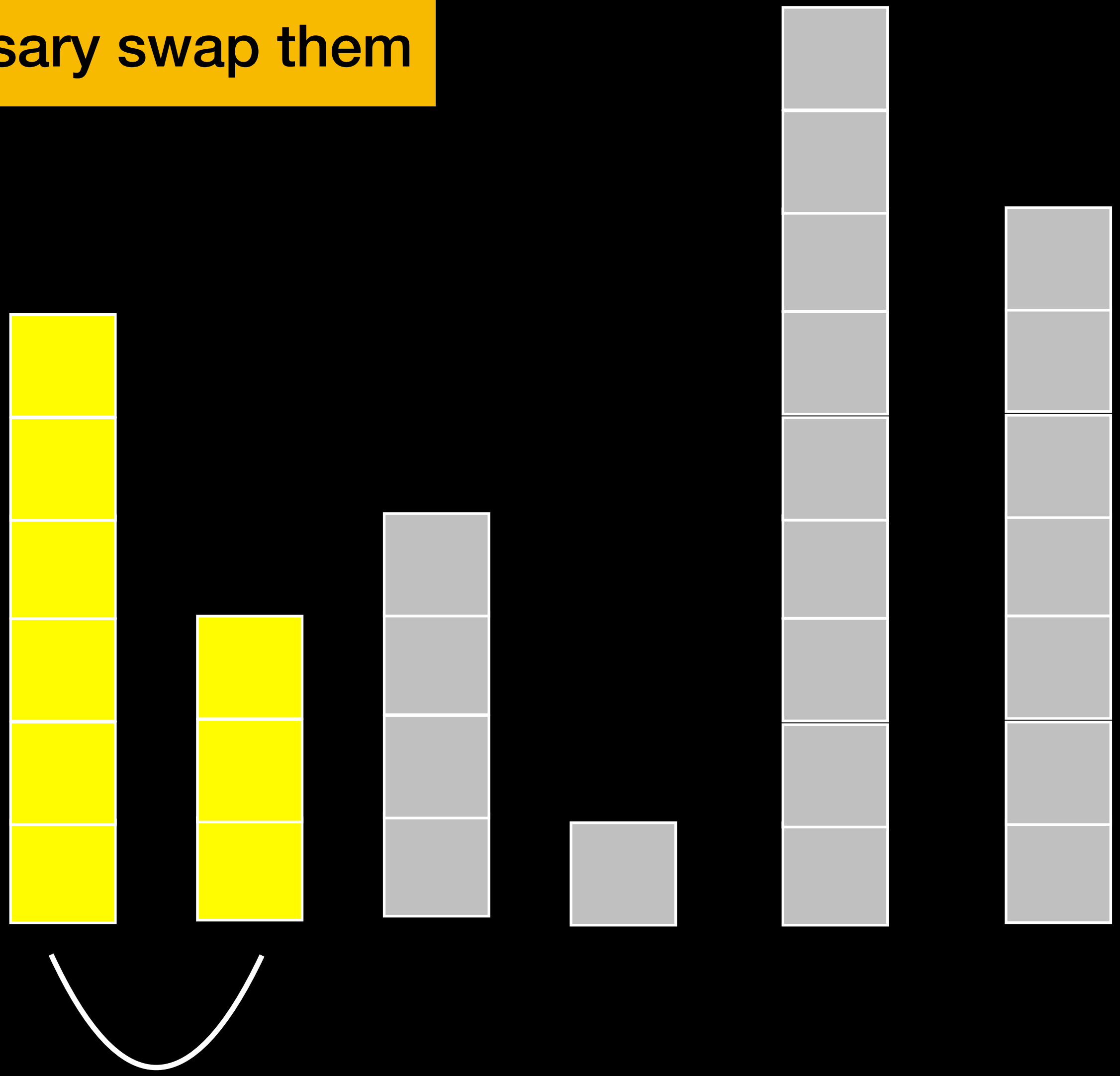
Bubble Sort

▨ Unsorted

🟦 Sorted

Compare adjacent elements and if necessary swap them

1st Pass



Bubble Sort

▨ Unsorted

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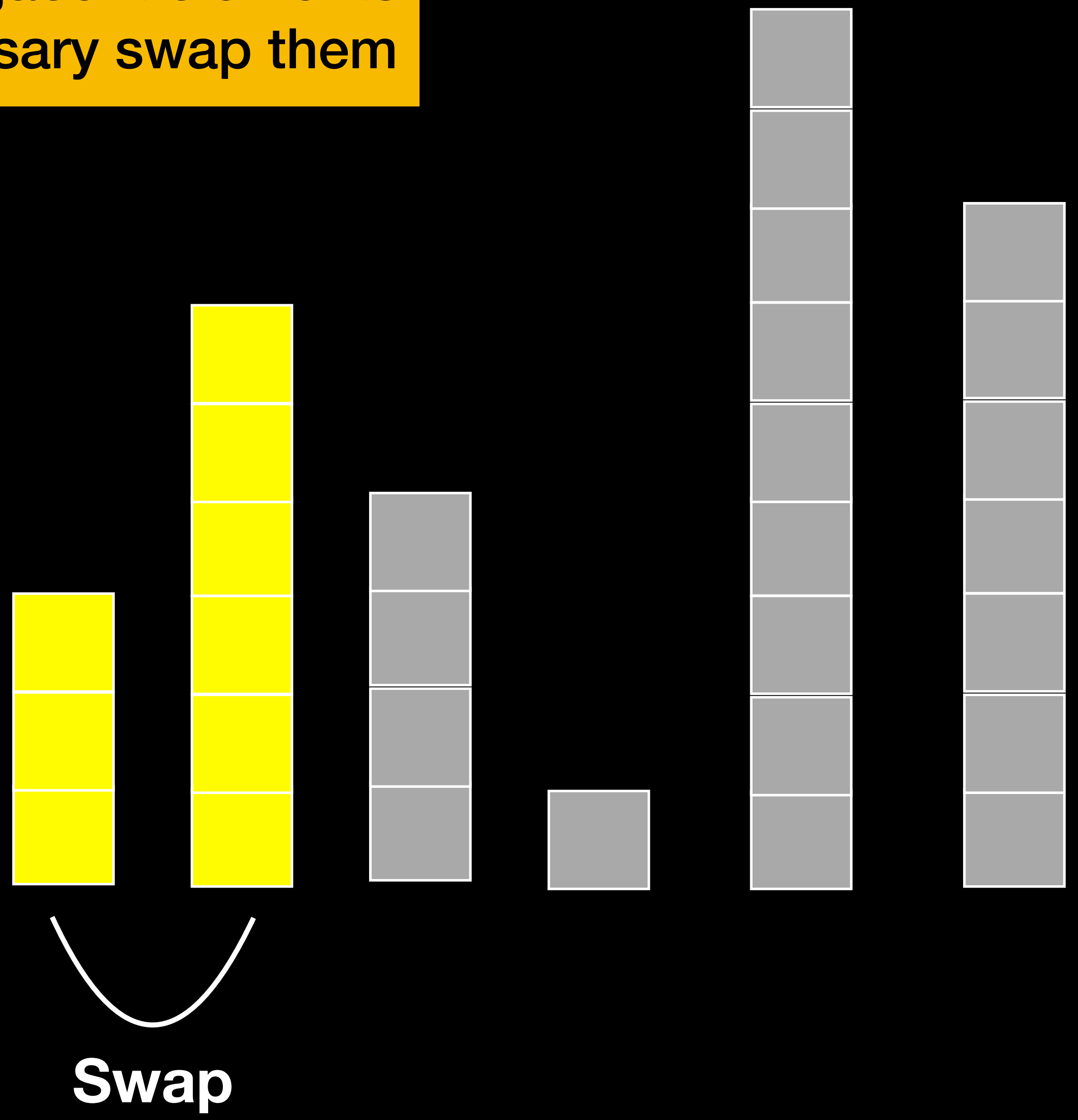
Compare adjacent elements and if necessary swap them

1st Pass



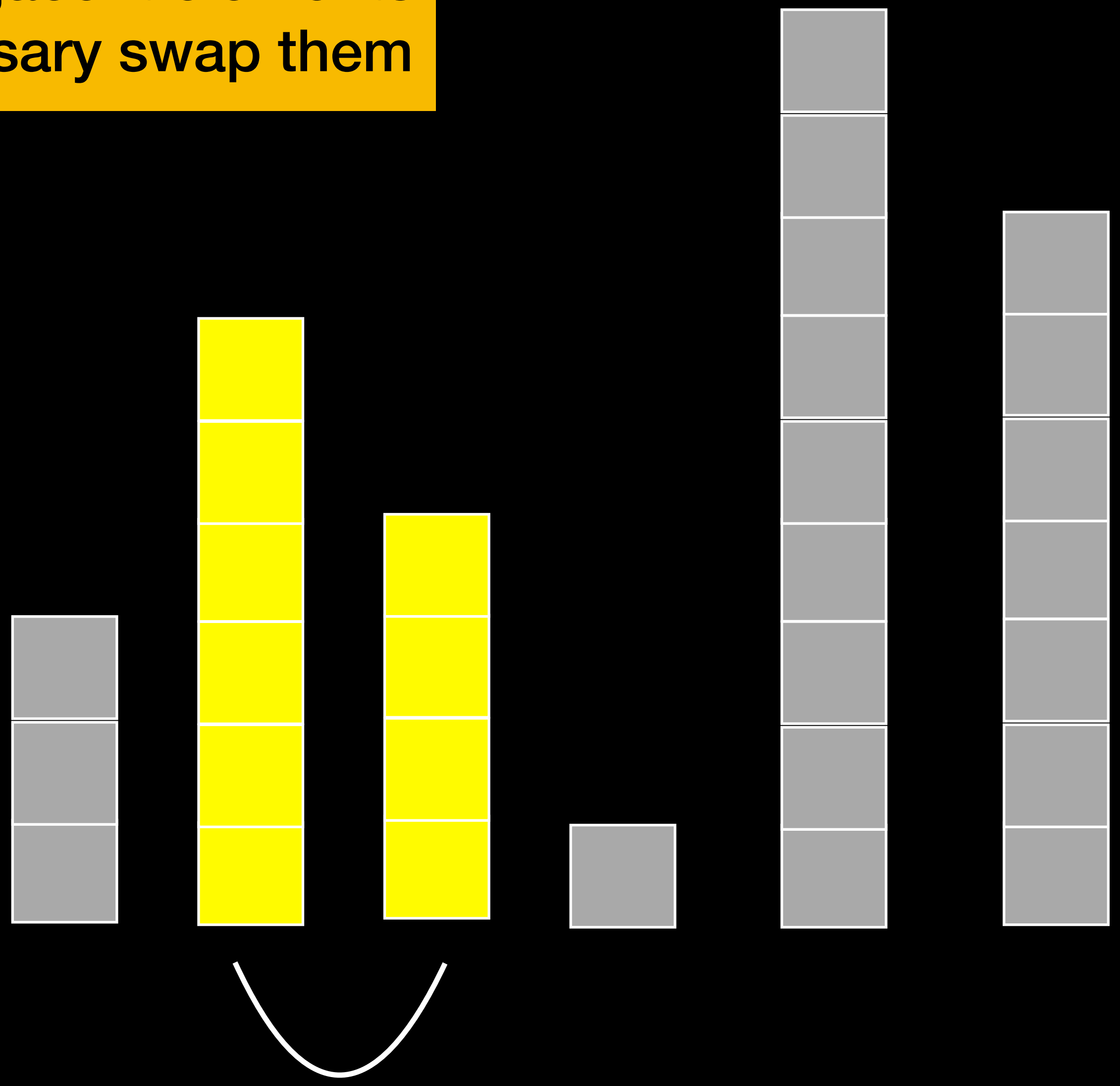
Bubble Sort

⬜ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

1st Pass



Bubble Sort

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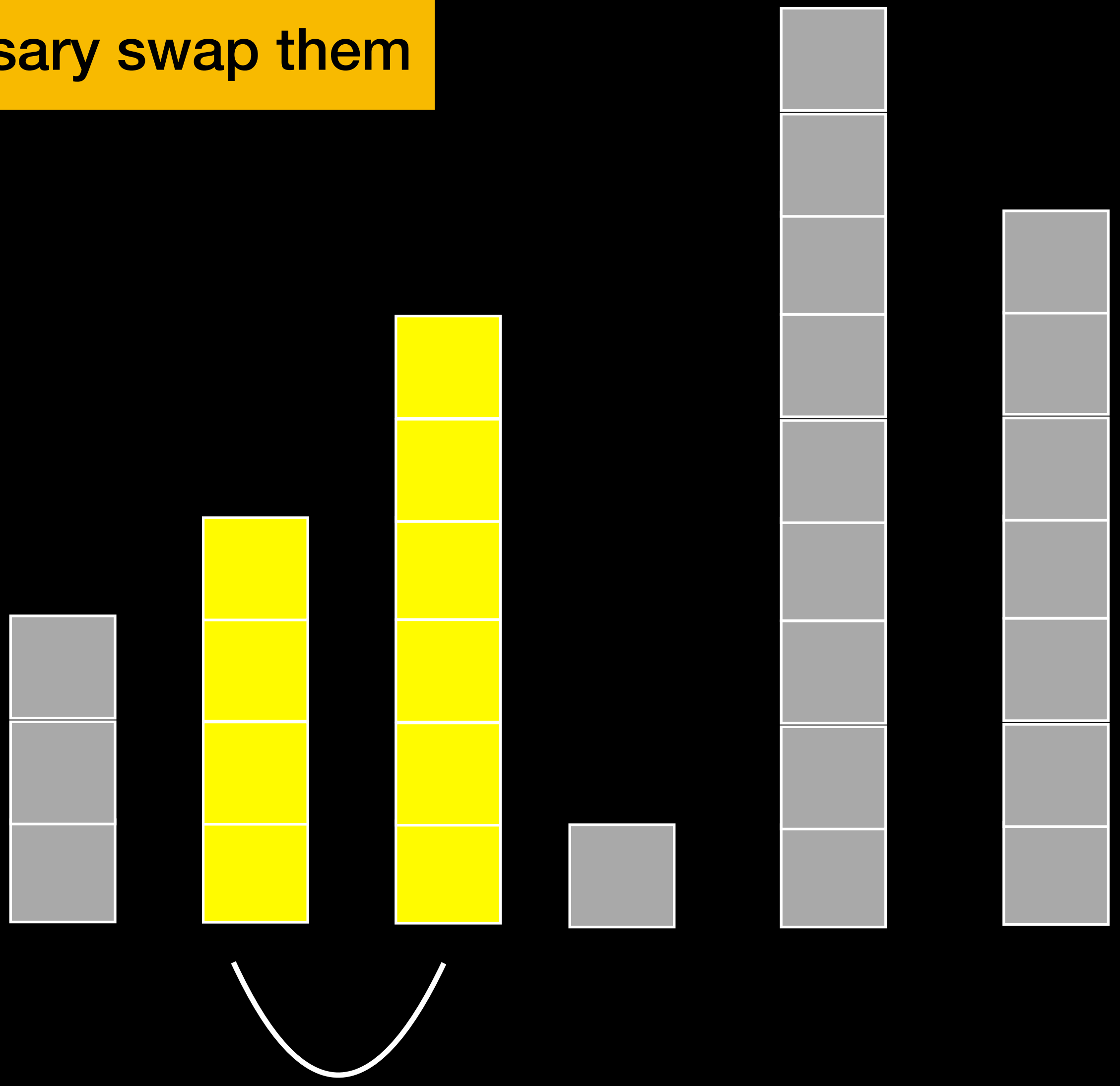


Swap

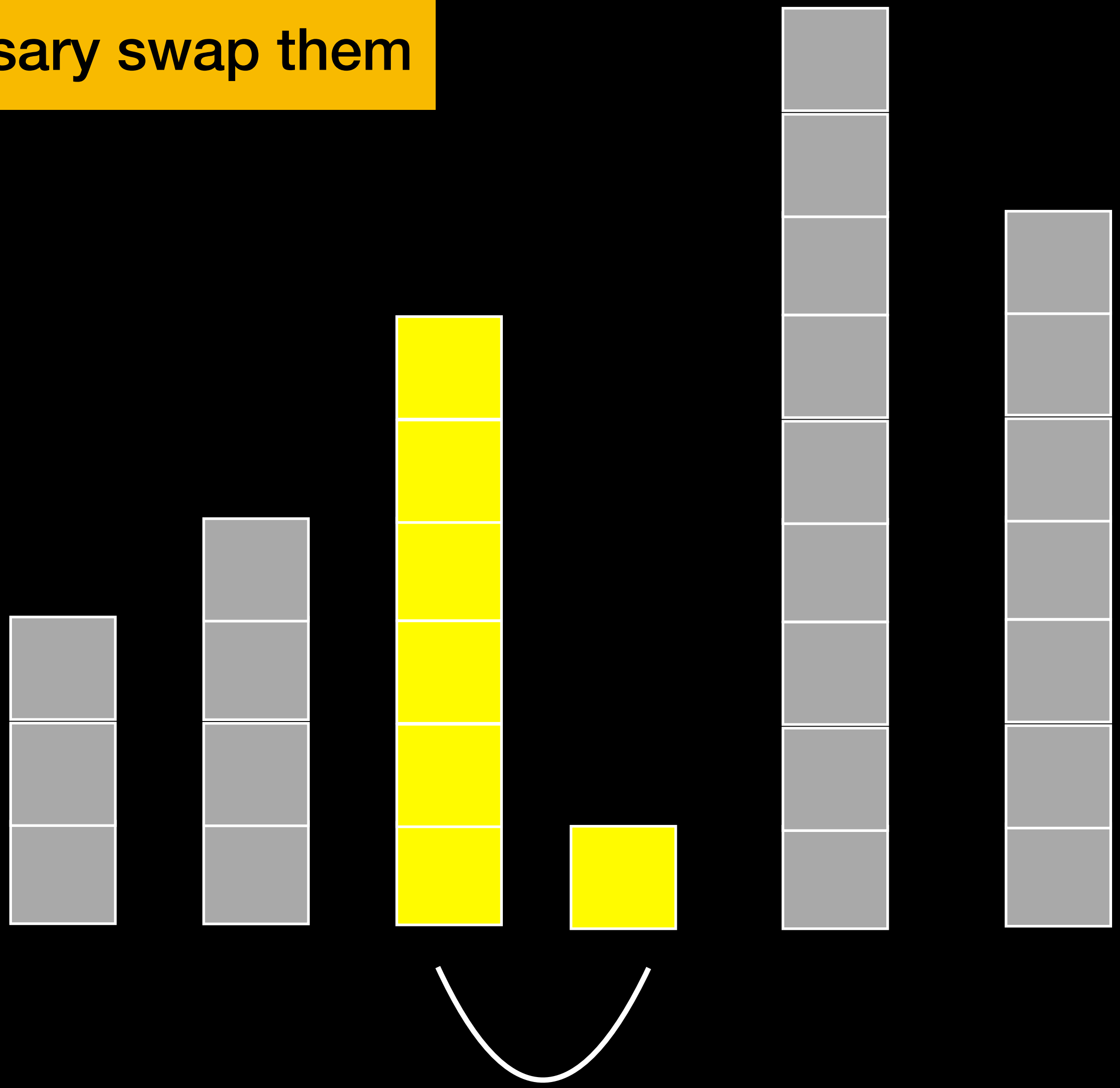
Bubble Sort

▨ Unsorted

🟦 Sorted

Compare adjacent elements and if necessary swap them

1st Pass



Bubble Sort

▨ Unsorted

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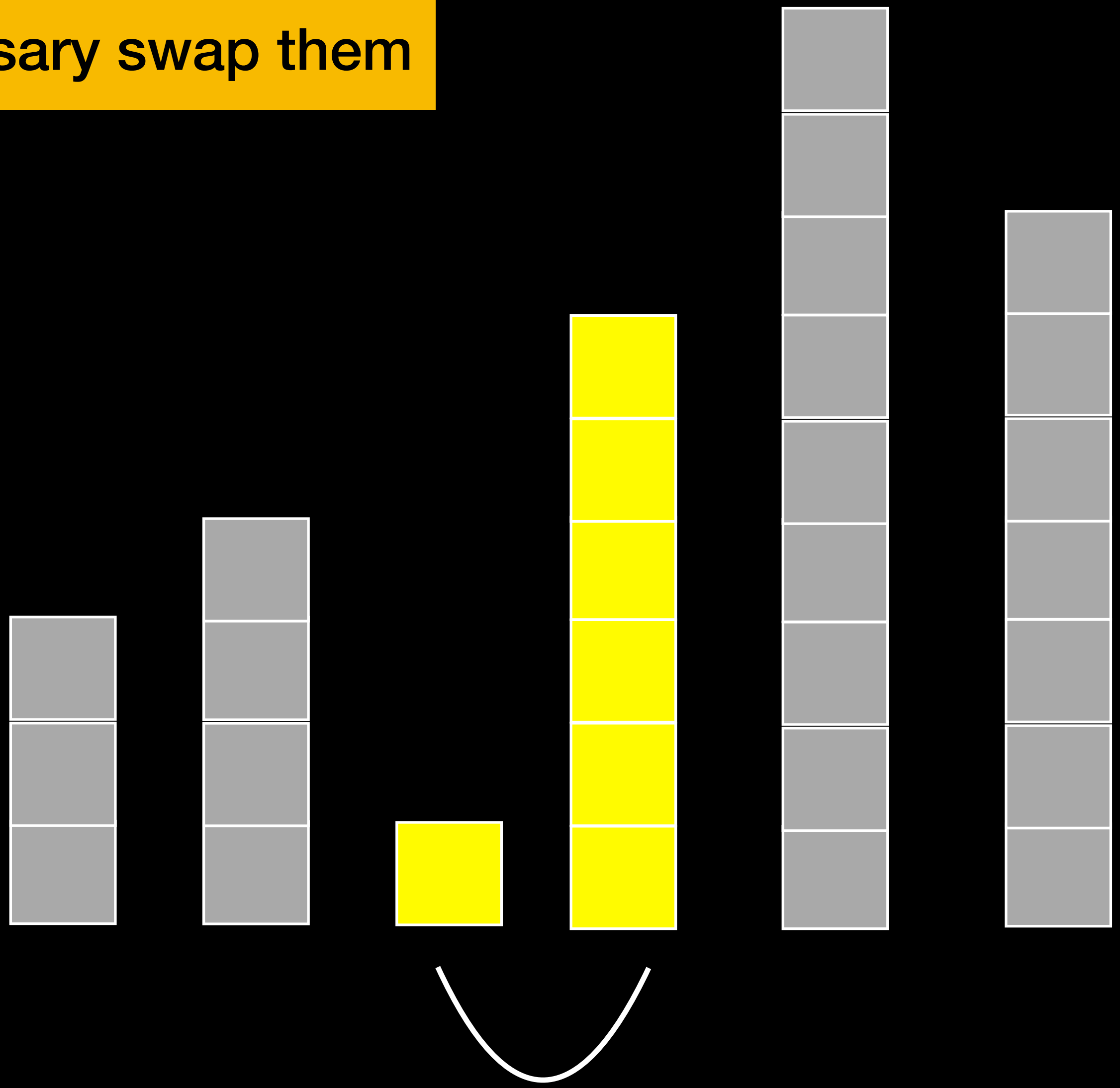
Compare adjacent elements and if necessary swap them

1st Pass



Swap

Bubble Sort

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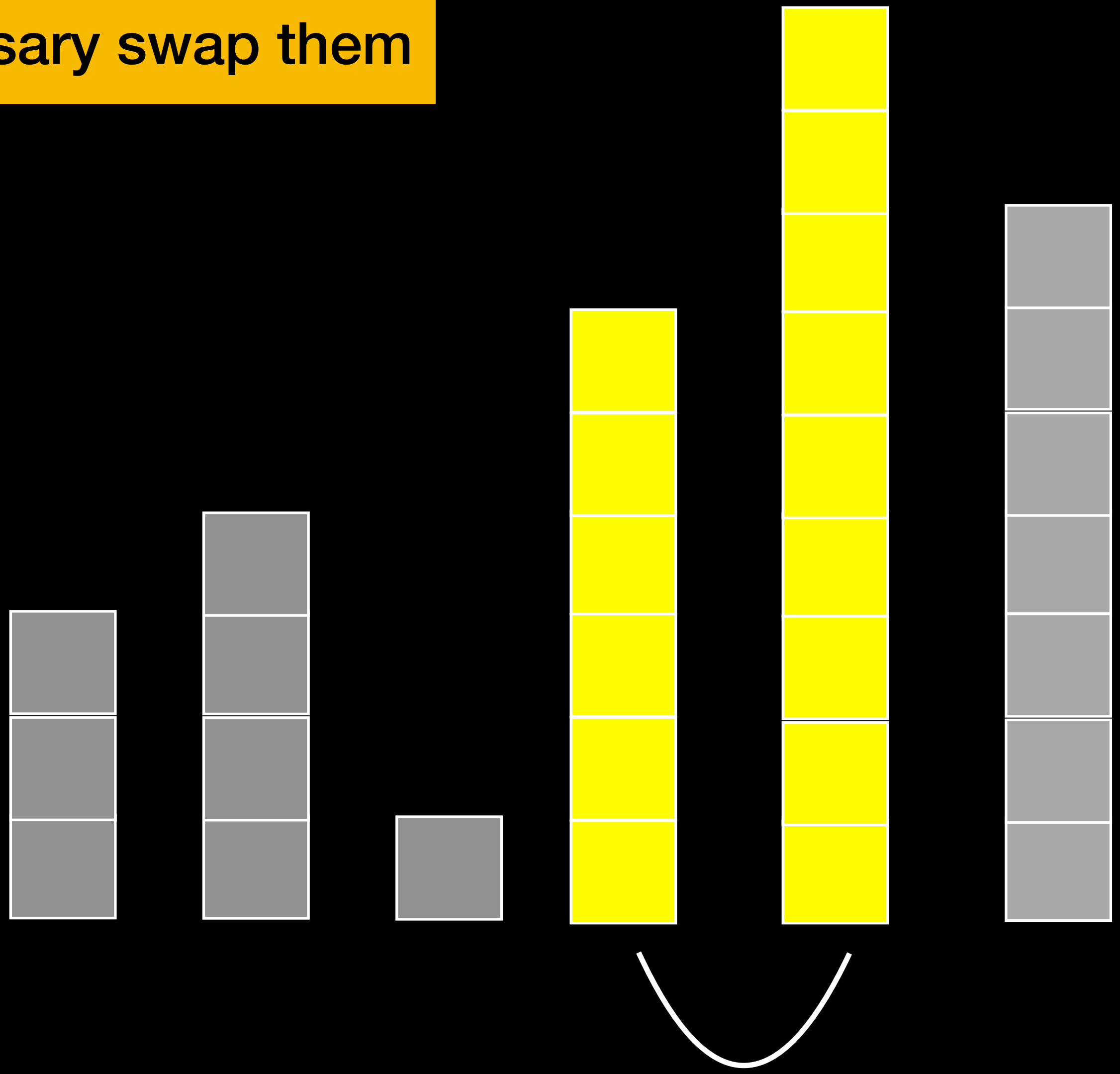
1st Pass



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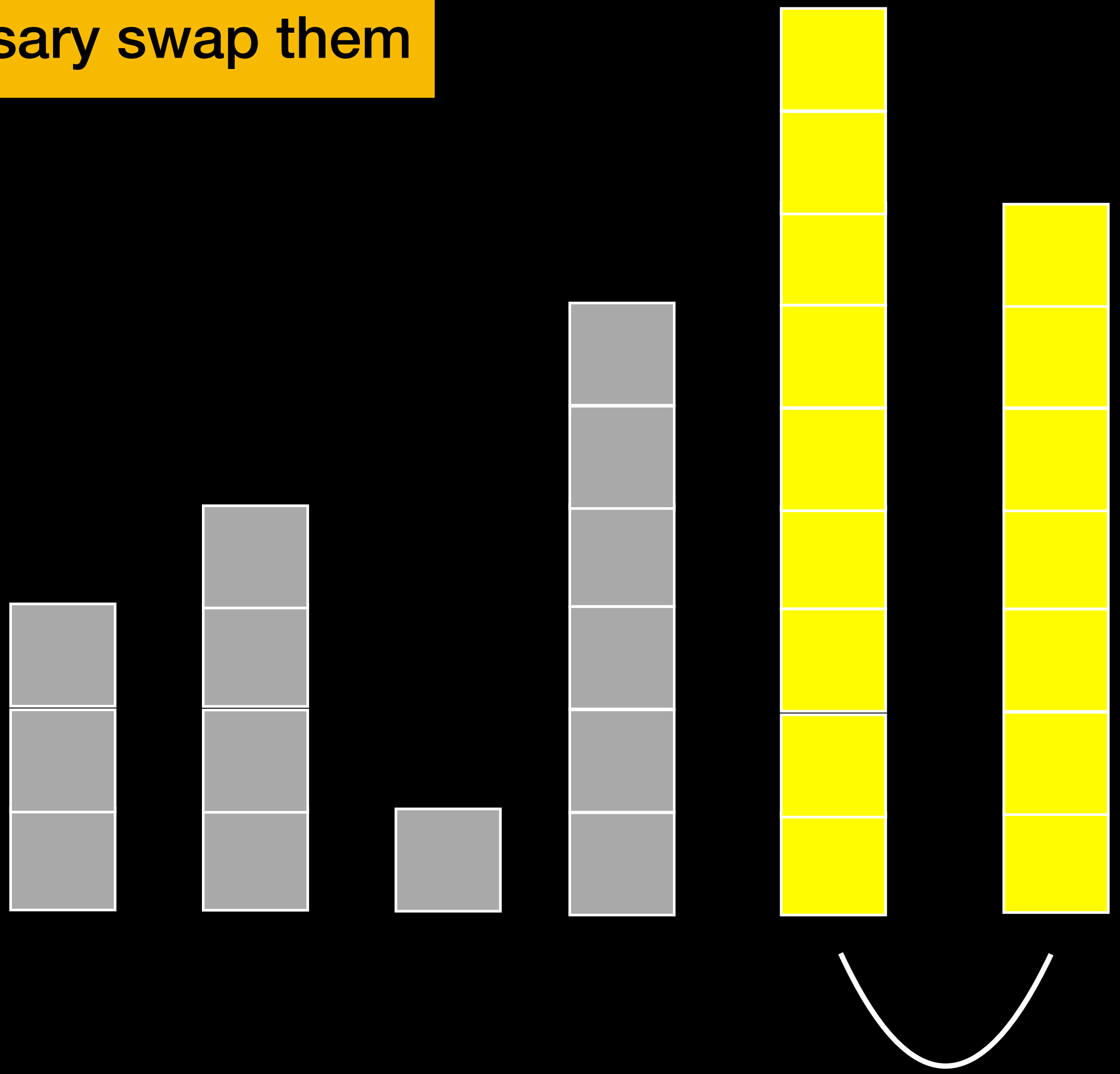

Bubble Sort

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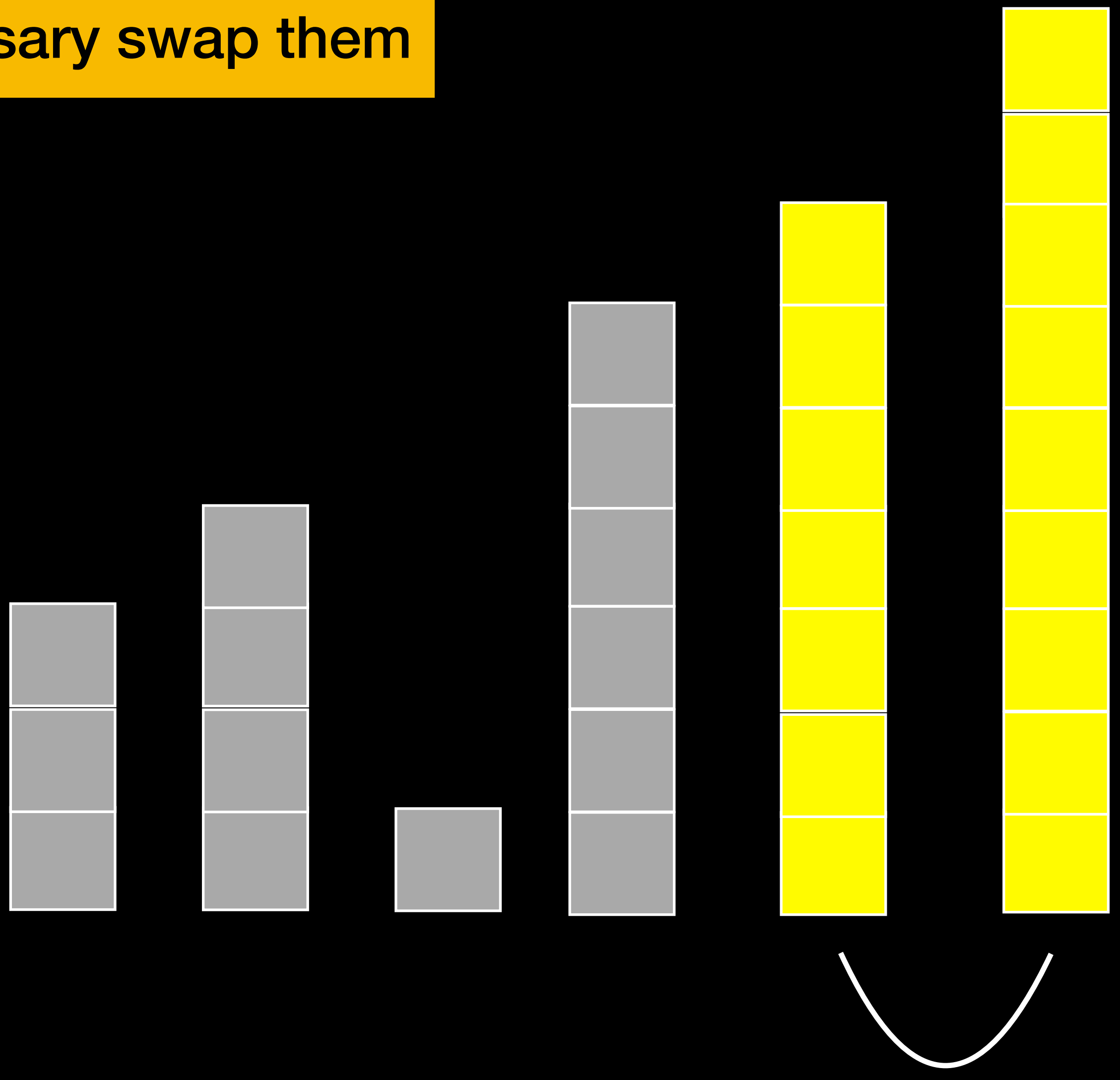
1st Pass

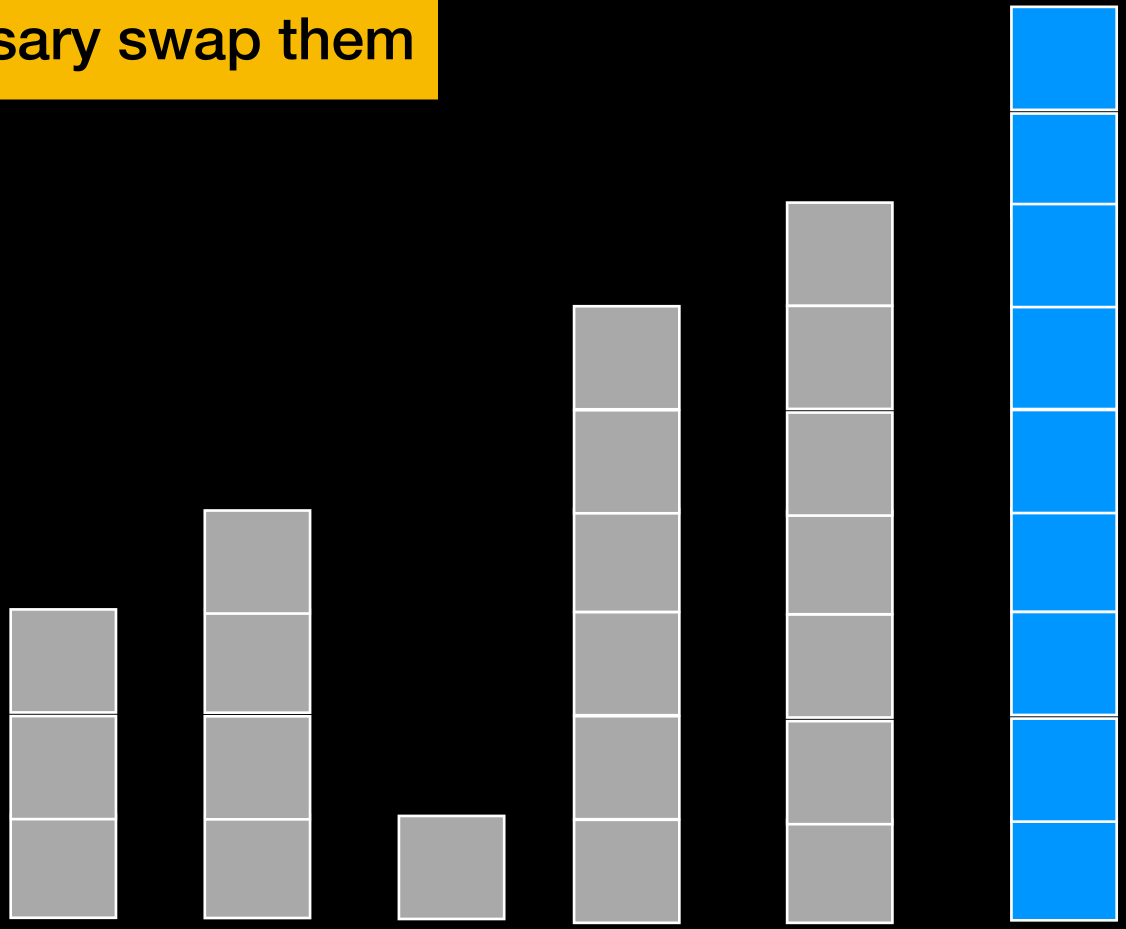


Swap

Bubble Sort

Compare adjacent elements and if necessary swap them

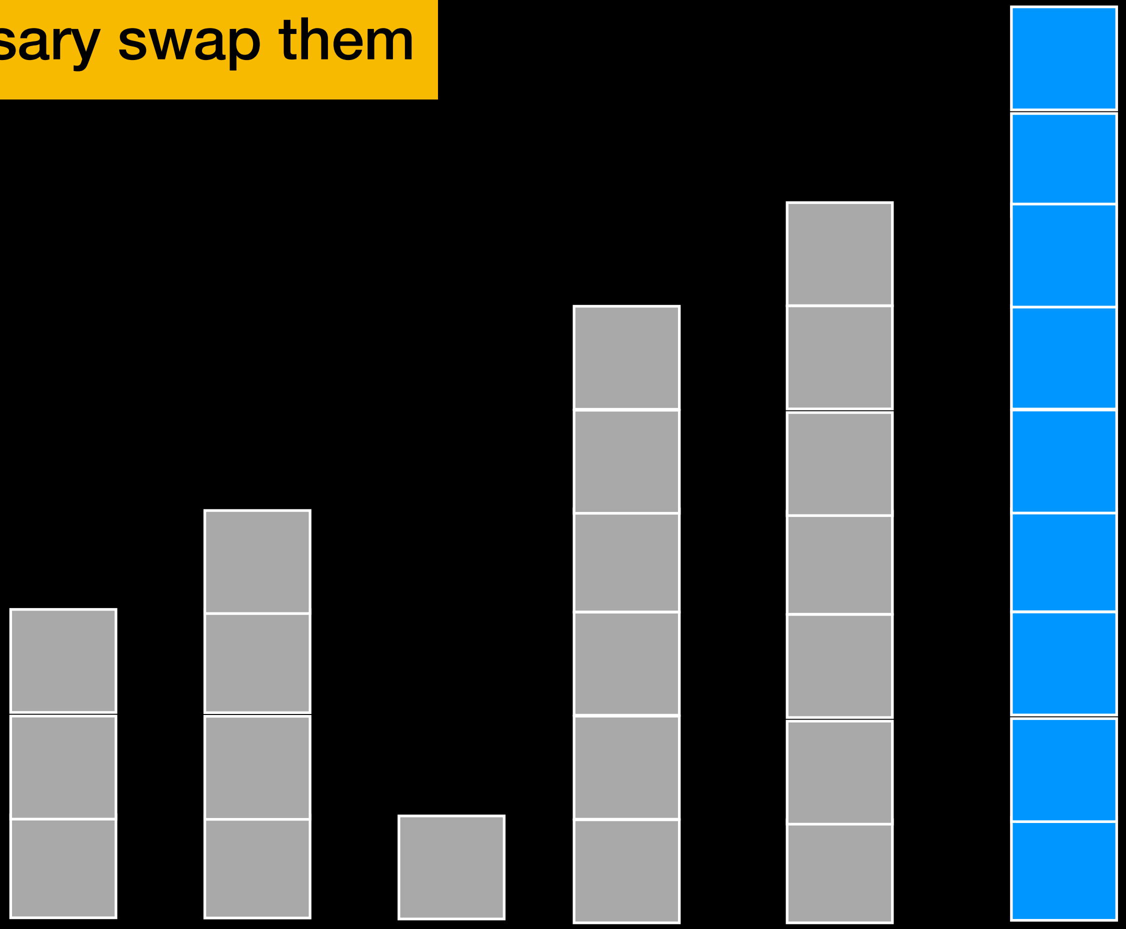
End of1st Pass:
Not sorted, but largest has ***"bubbled up"*** to its proper position



Bubble Sort

Compare adjacent elements and if necessary swap them

2nd Pass:
Sort **n-1**



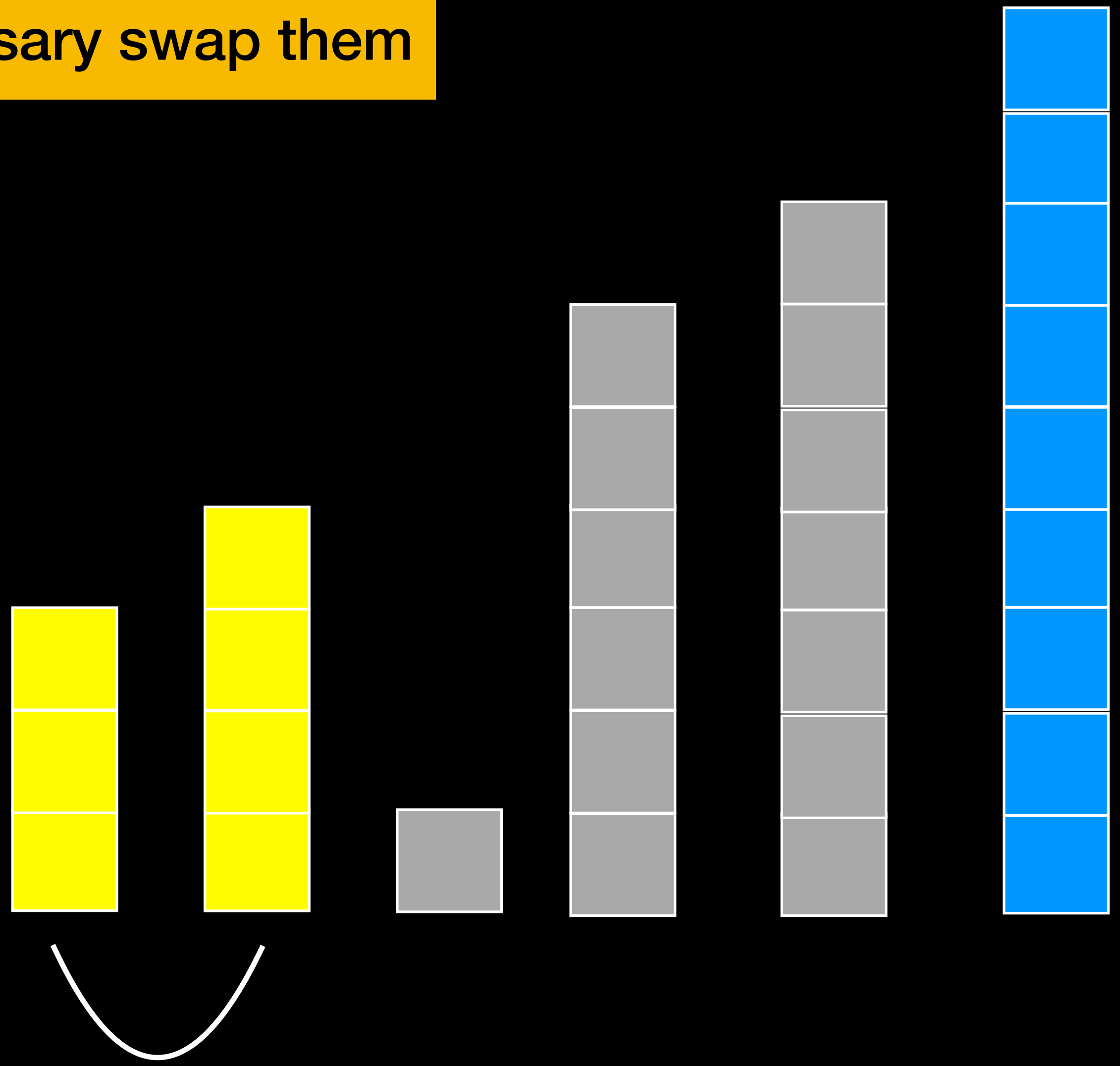
Bubble Sort

▨ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

2nd Pass



Bubble Sort

⬜ **Unsorted**

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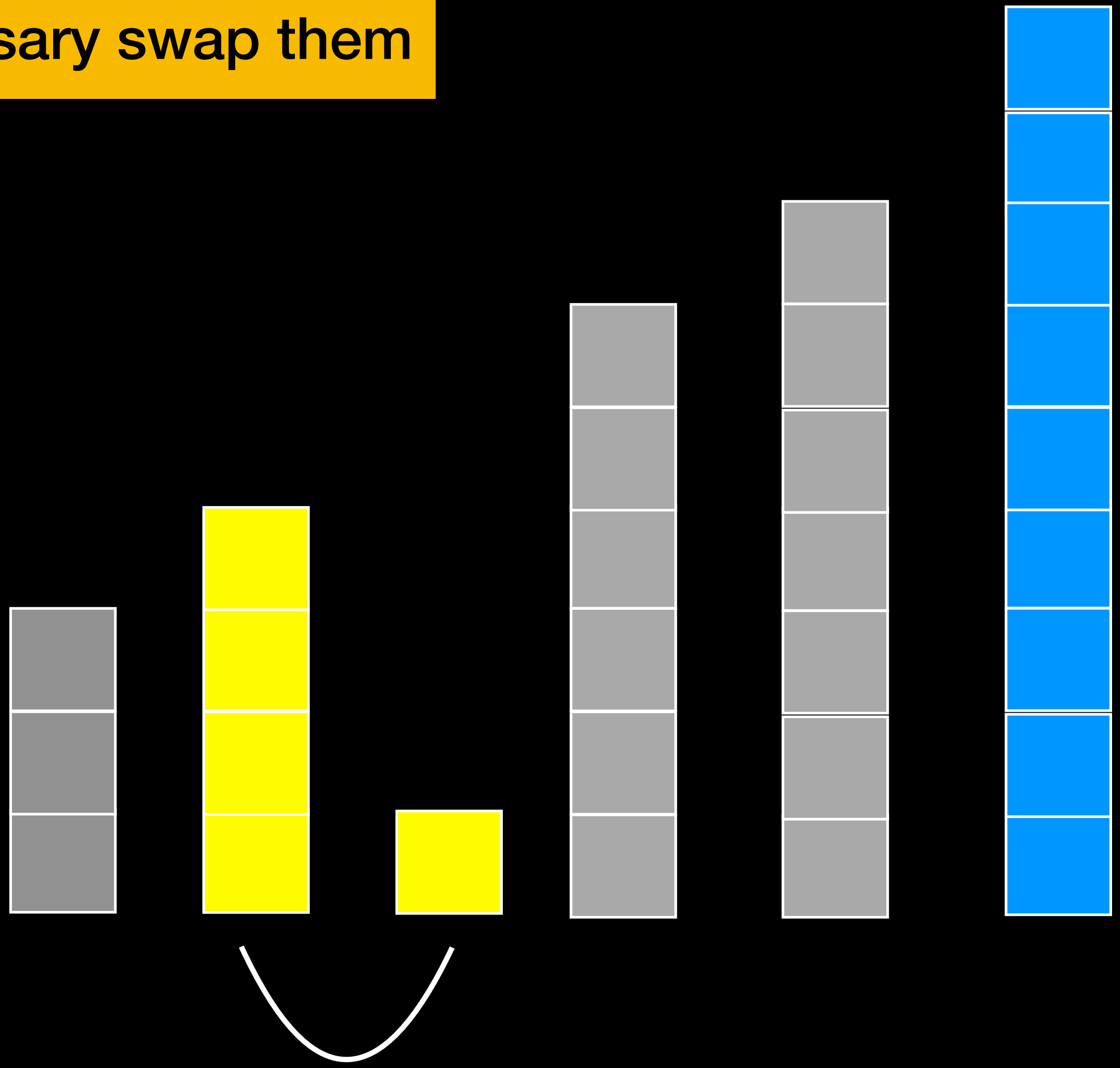
Compare adjacent elements and if necessary swap them

2nd Pass



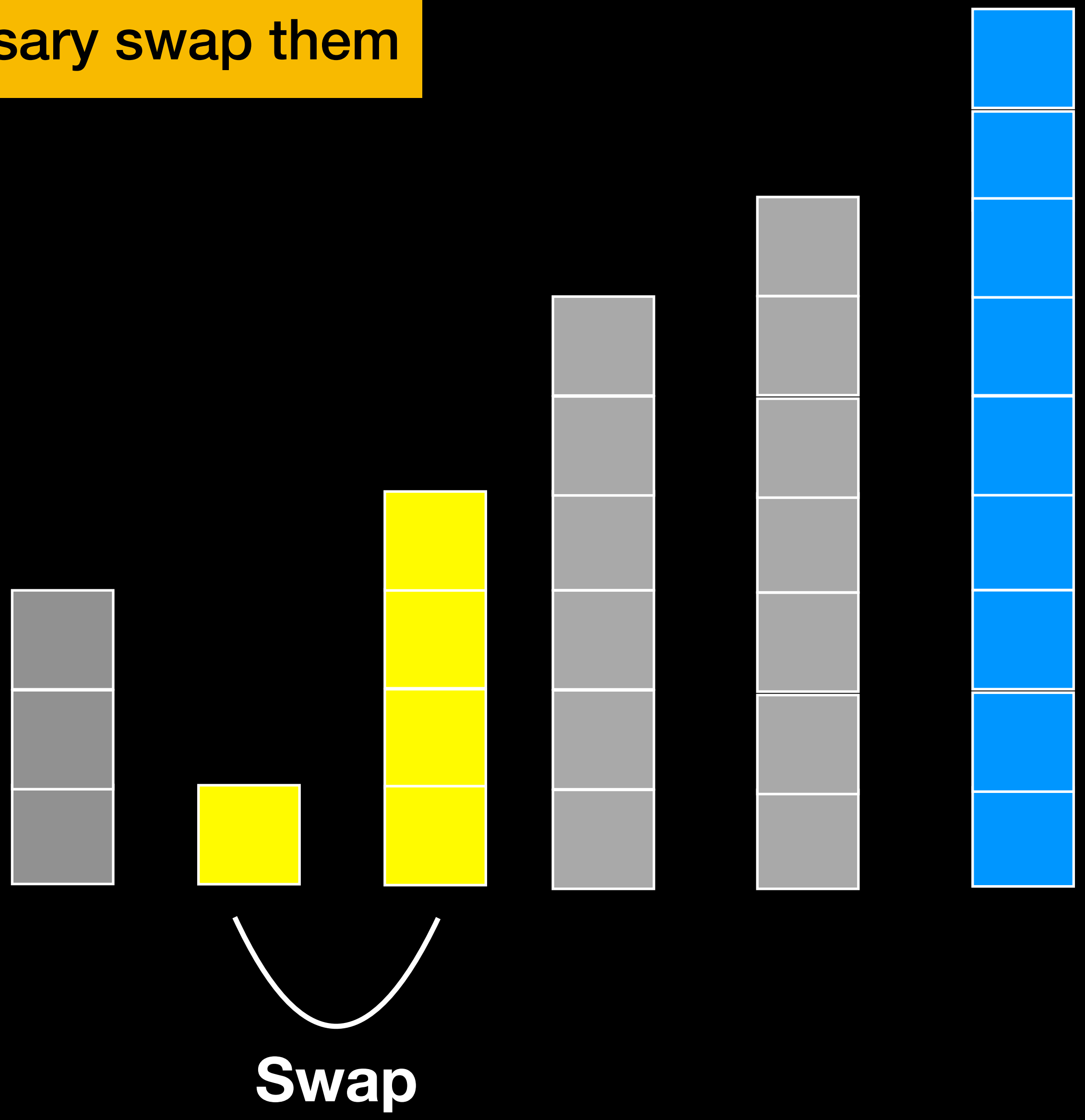
Bubble Sort

▨ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

2nd Pass



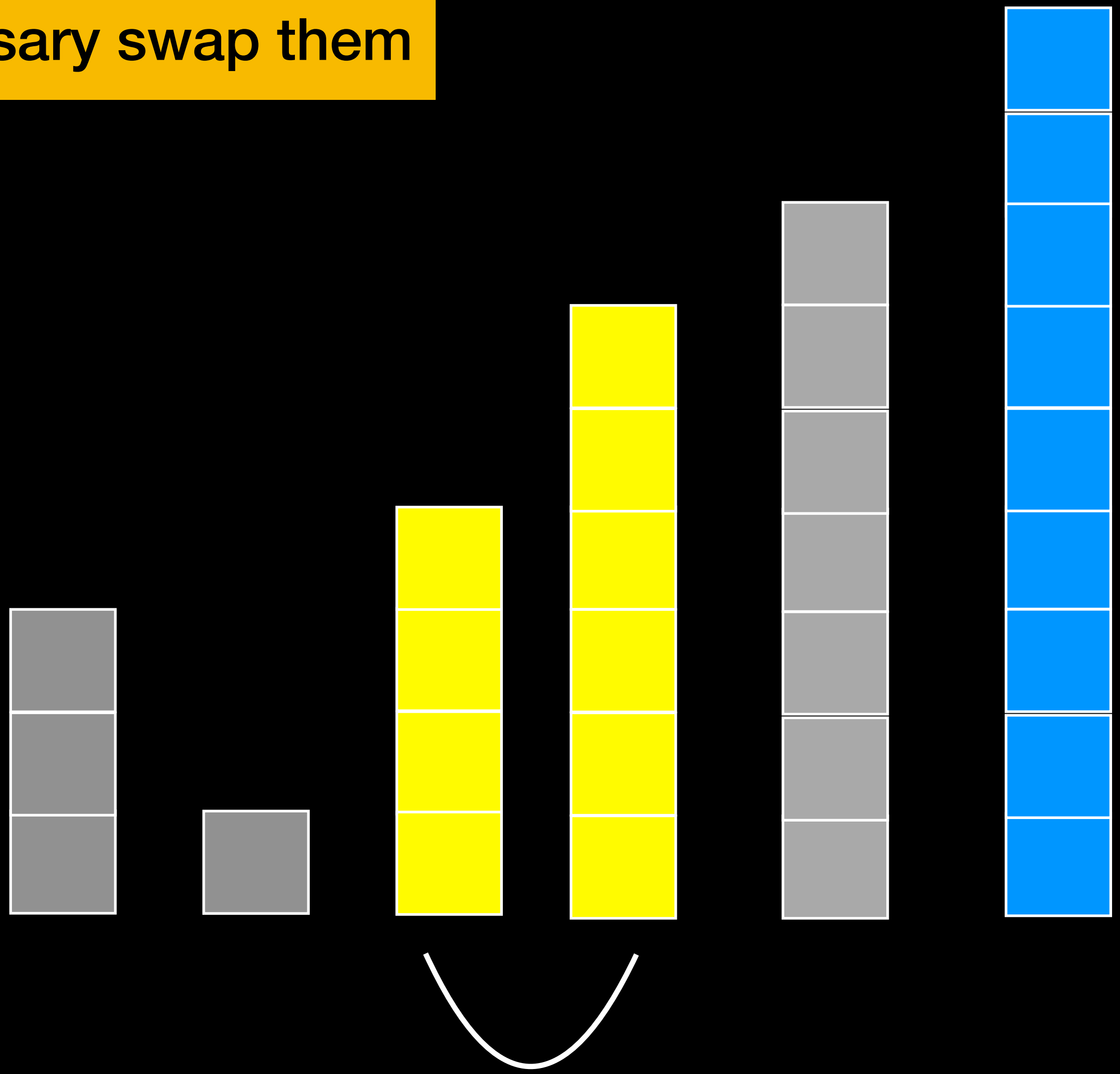
Bubble Sort

▨ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

2nd Pass

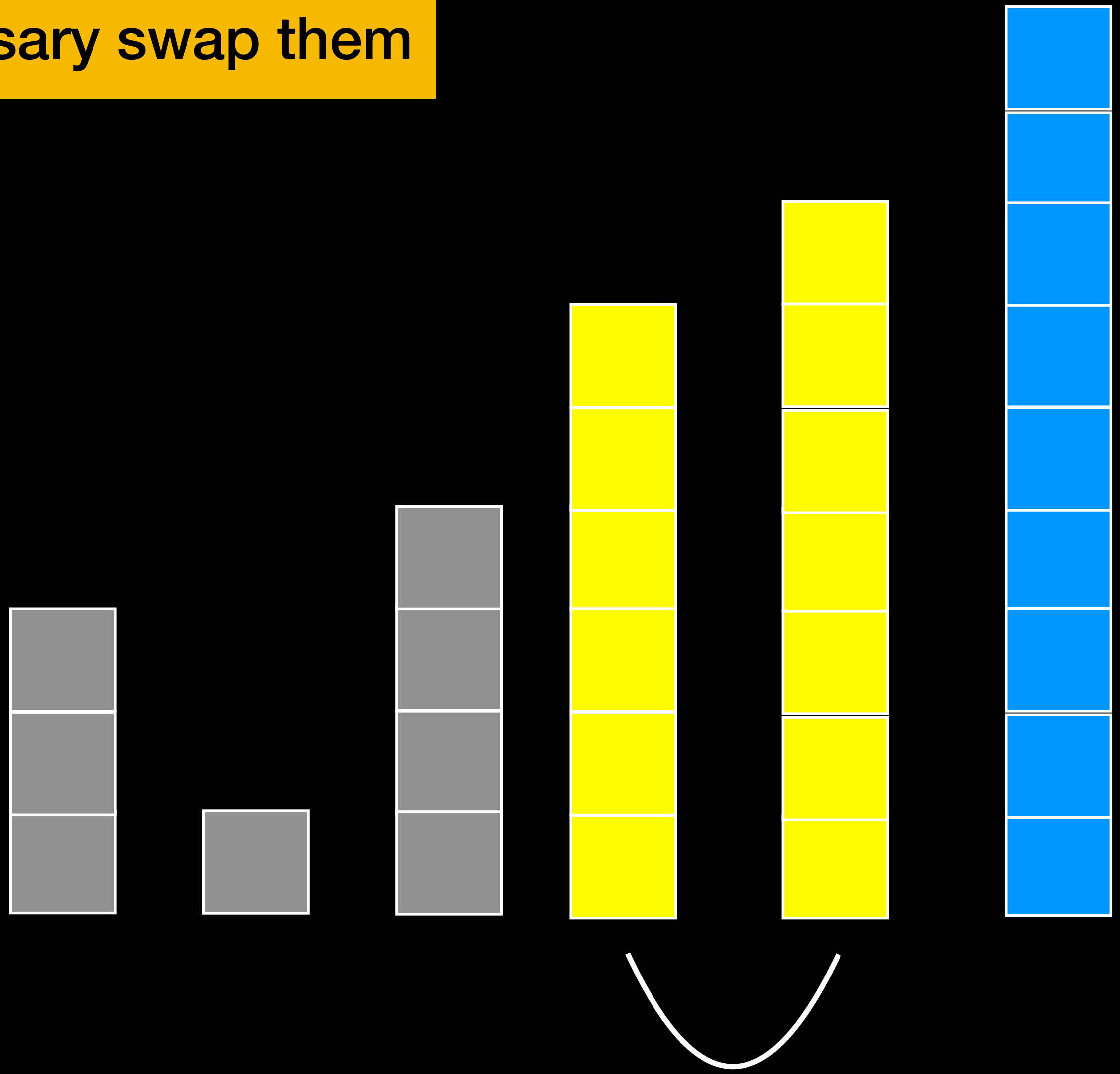


Bubble Sort

- ⬜ **Unsorted**
- 🟦 **Sorted**

Compare adjacent elements and if necessary swap them

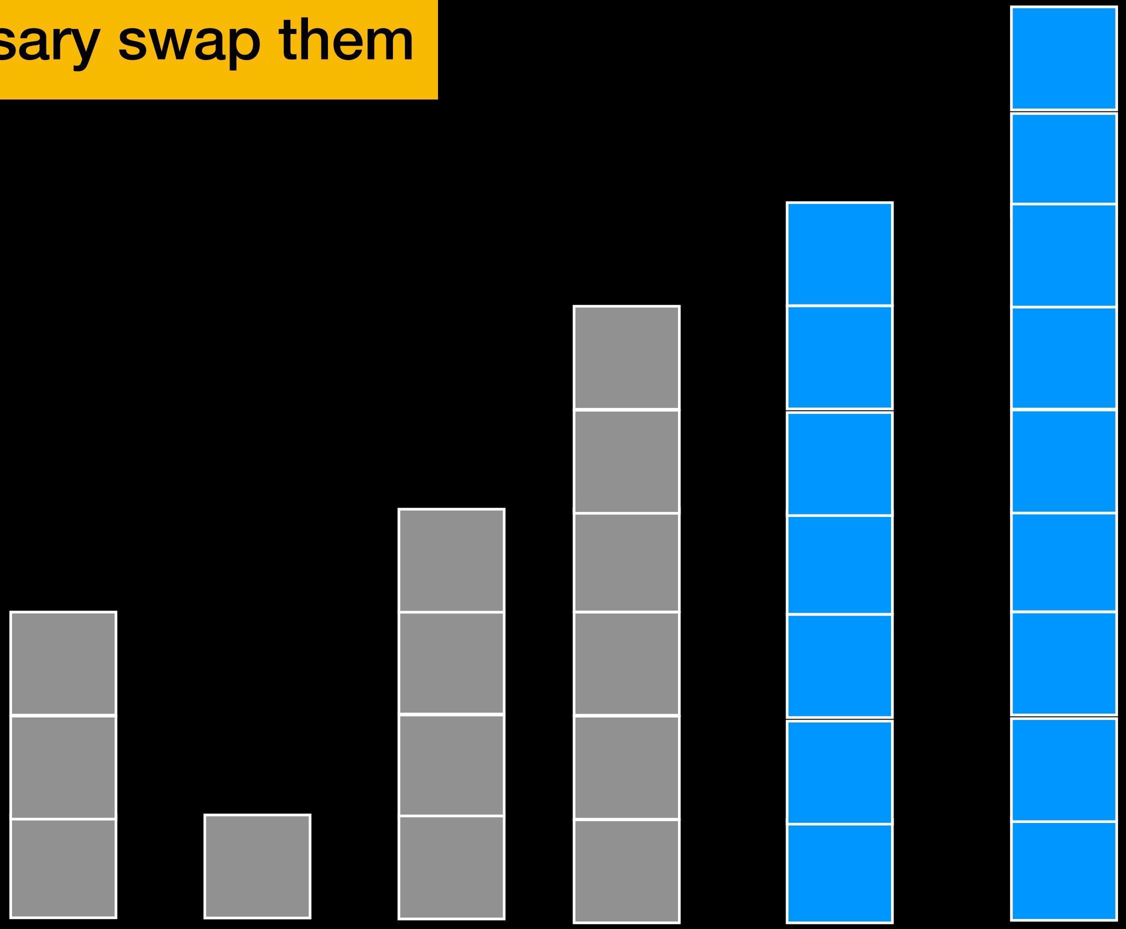
2nd Pass



Bubble Sort

Compare adjacent elements and if necessary swap them

3rd Pass:
Sort **n-2**



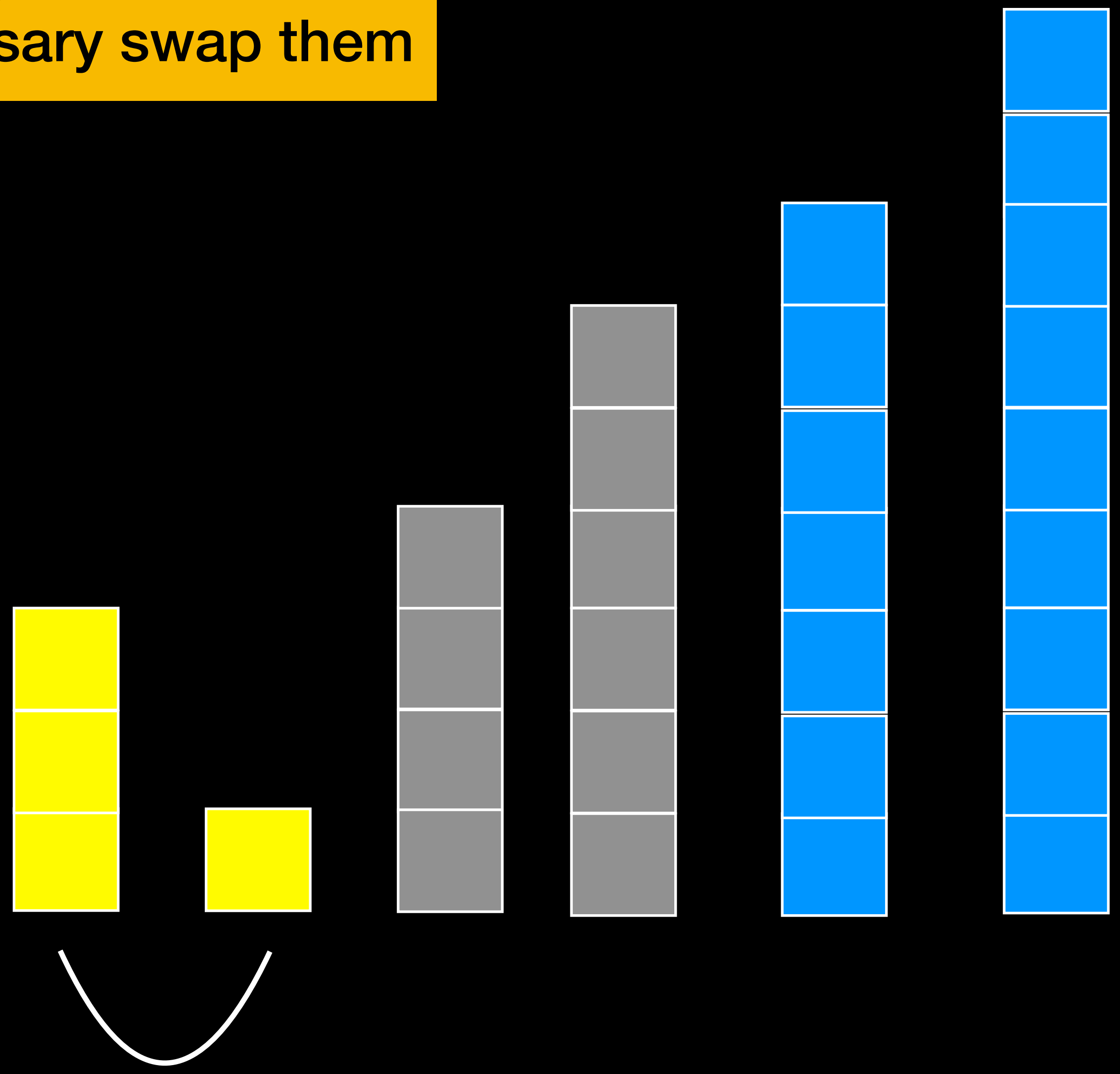
Bubble Sort

▨ Unsorted

🟦 Sorted

Compare adjacent elements and if necessary swap them

3rd Pass



Bubble Sort

◼ Unsorted

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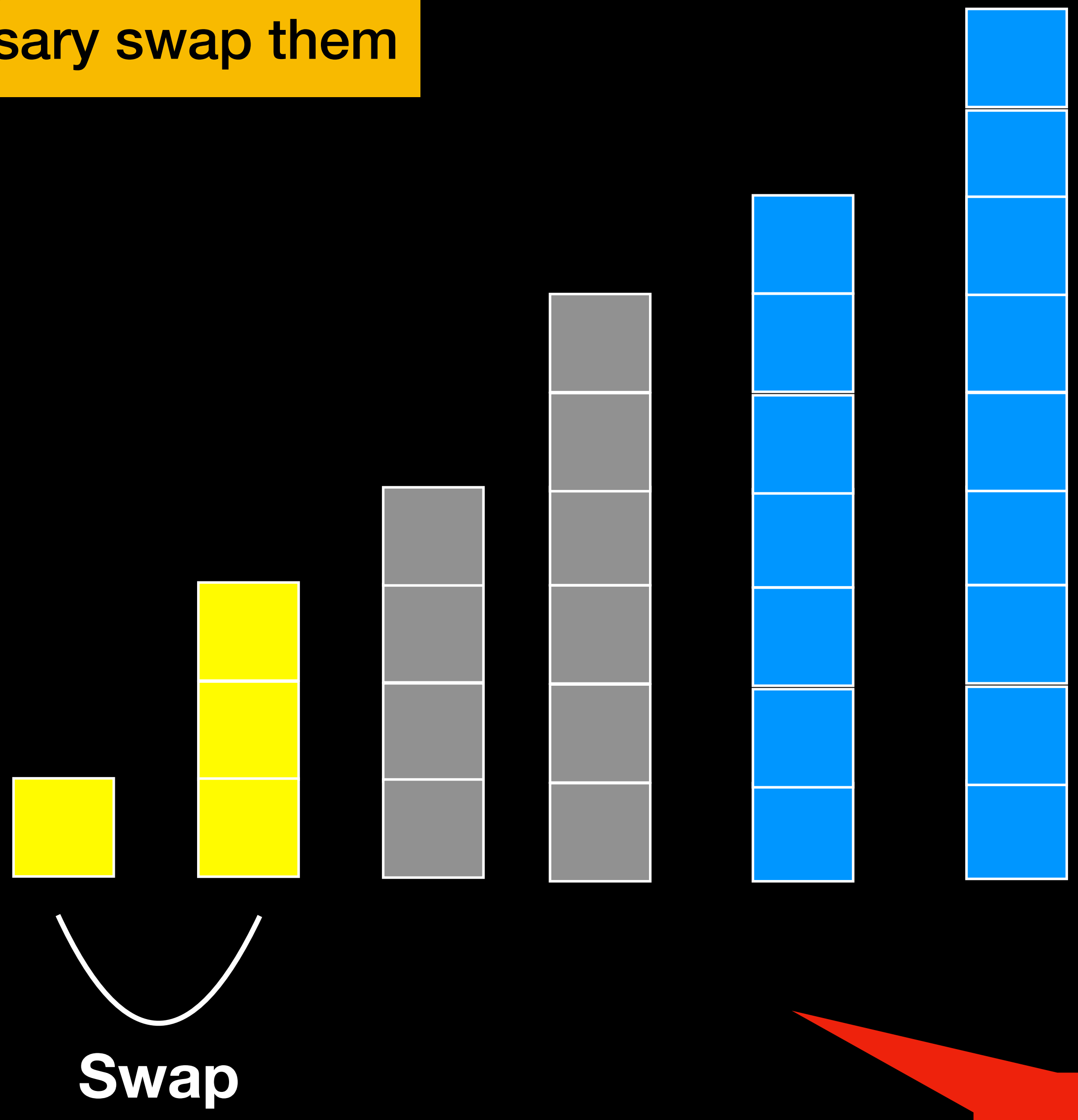
Compare adjacent elements and if necessary swap them

3rd Pass



Array is sorted
But our algorithm doesn’t know
It keeps on going

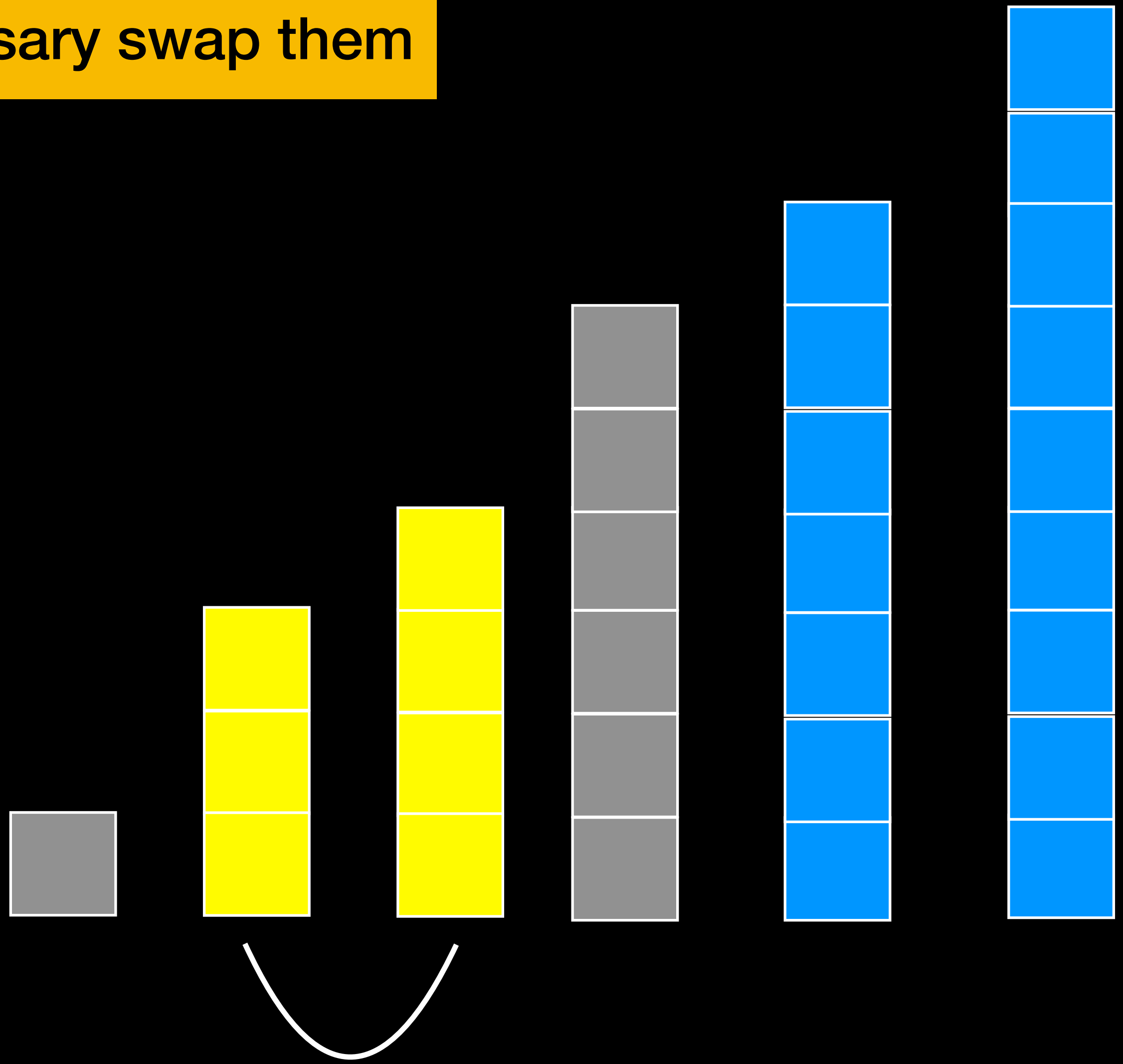
Bubble Sort

▨ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

3rd Pass



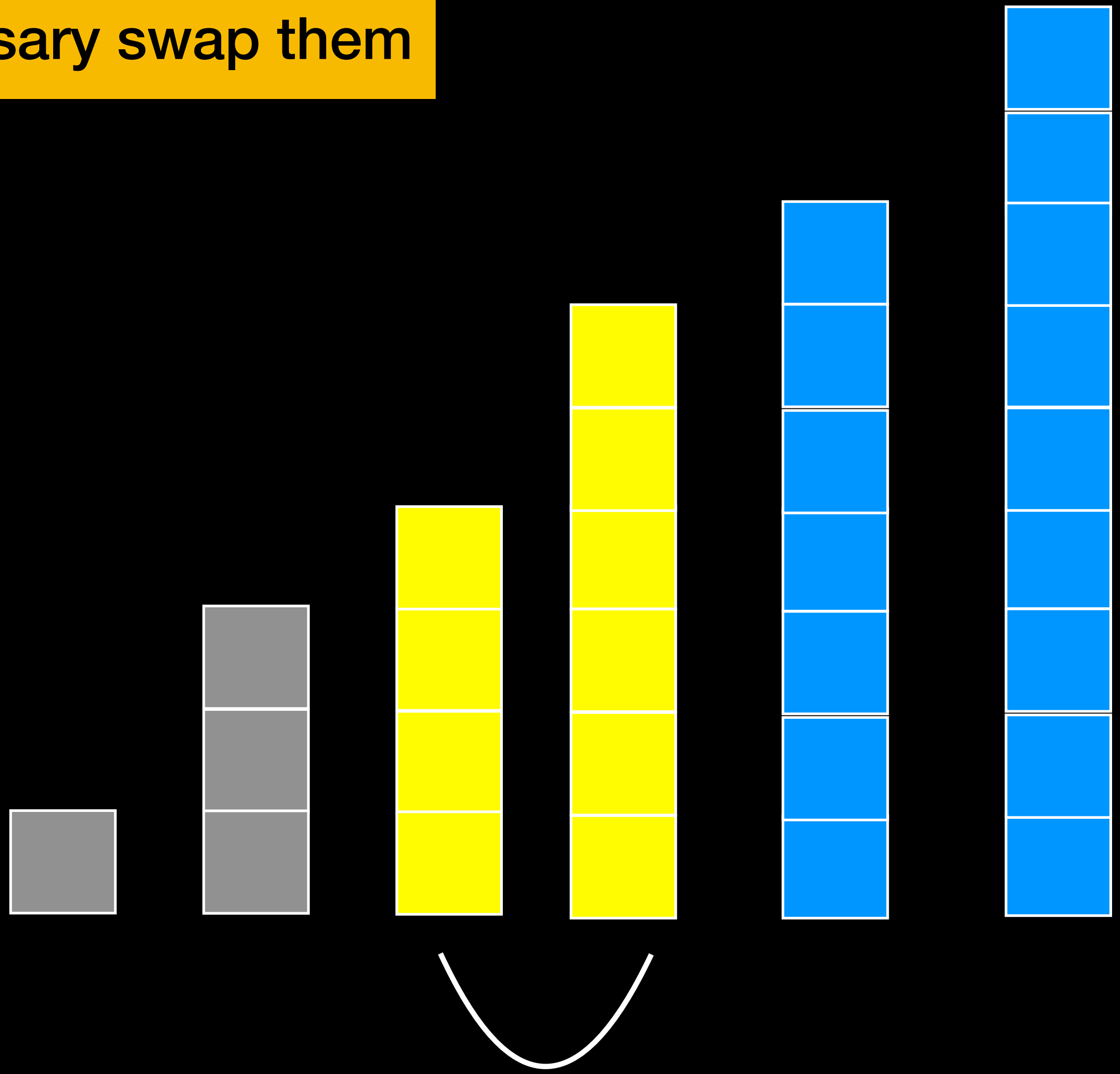
Bubble Sort

▨ Unsorted

🟦 Sorted

3rd Pass

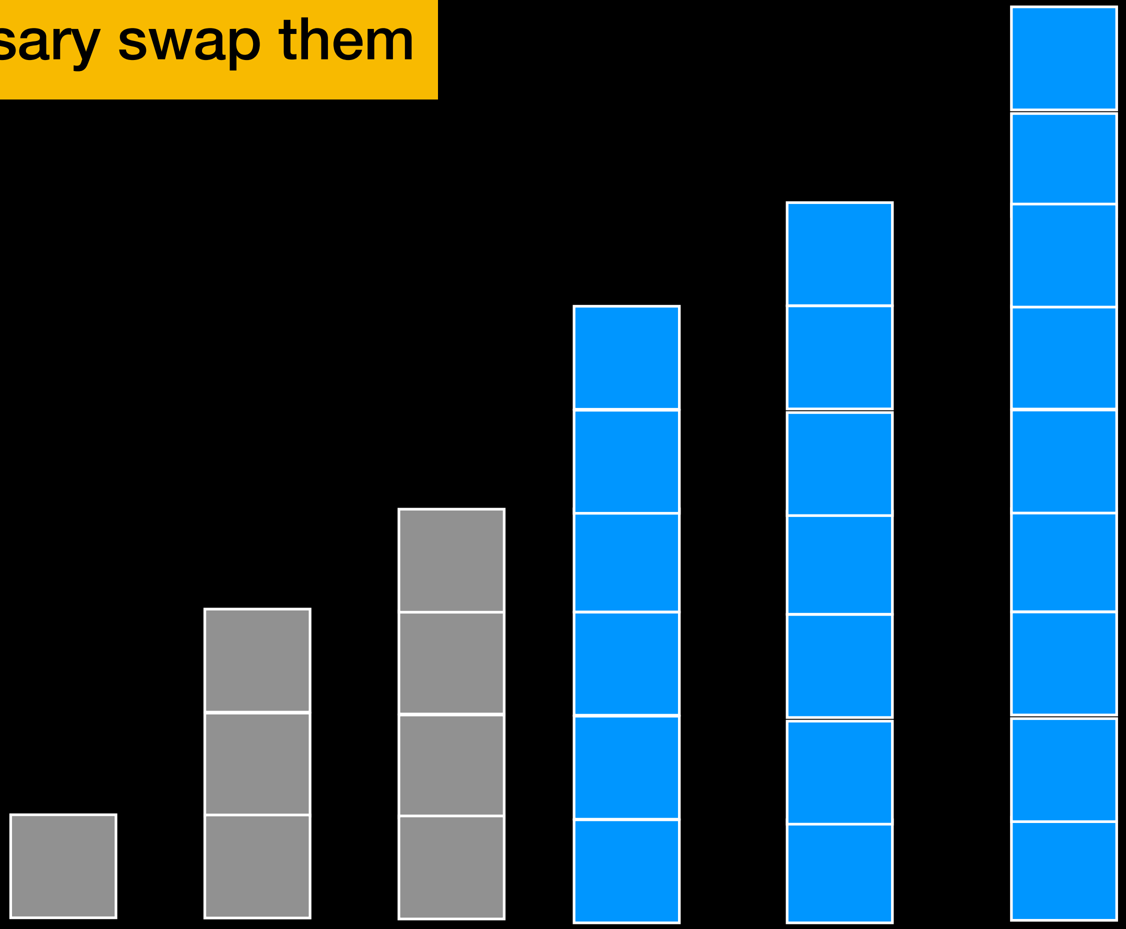
Compare adjacent elements and if necessary swap them



Bubble Sort

Compare adjacent elements and if necessary swap them

4th Pass:
Sort **n-3**



Bubble Sort

▨ Unsorted

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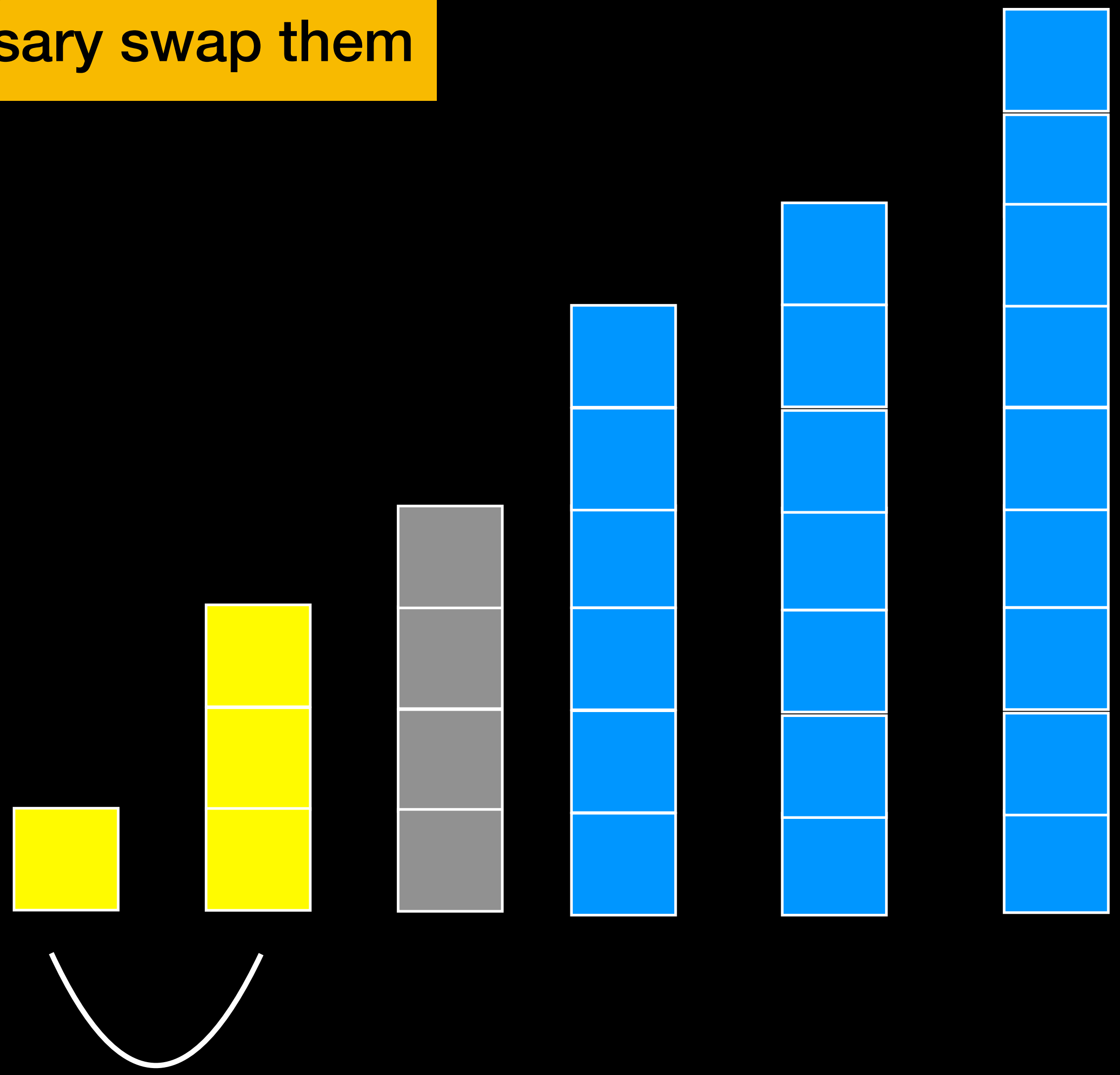
Compare adjacent elements and if necessary swap them

4th Pass



Bubble Sort

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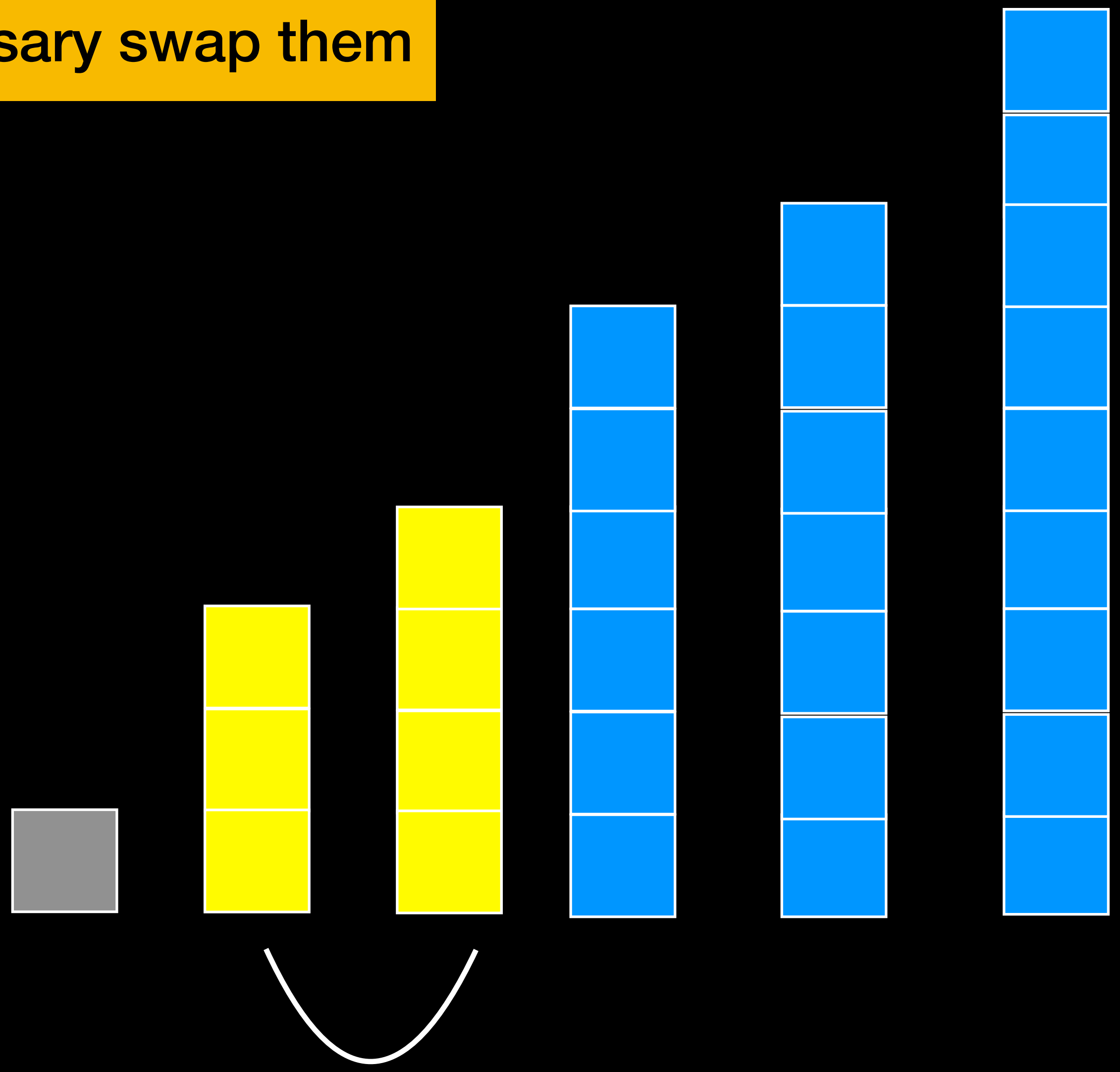
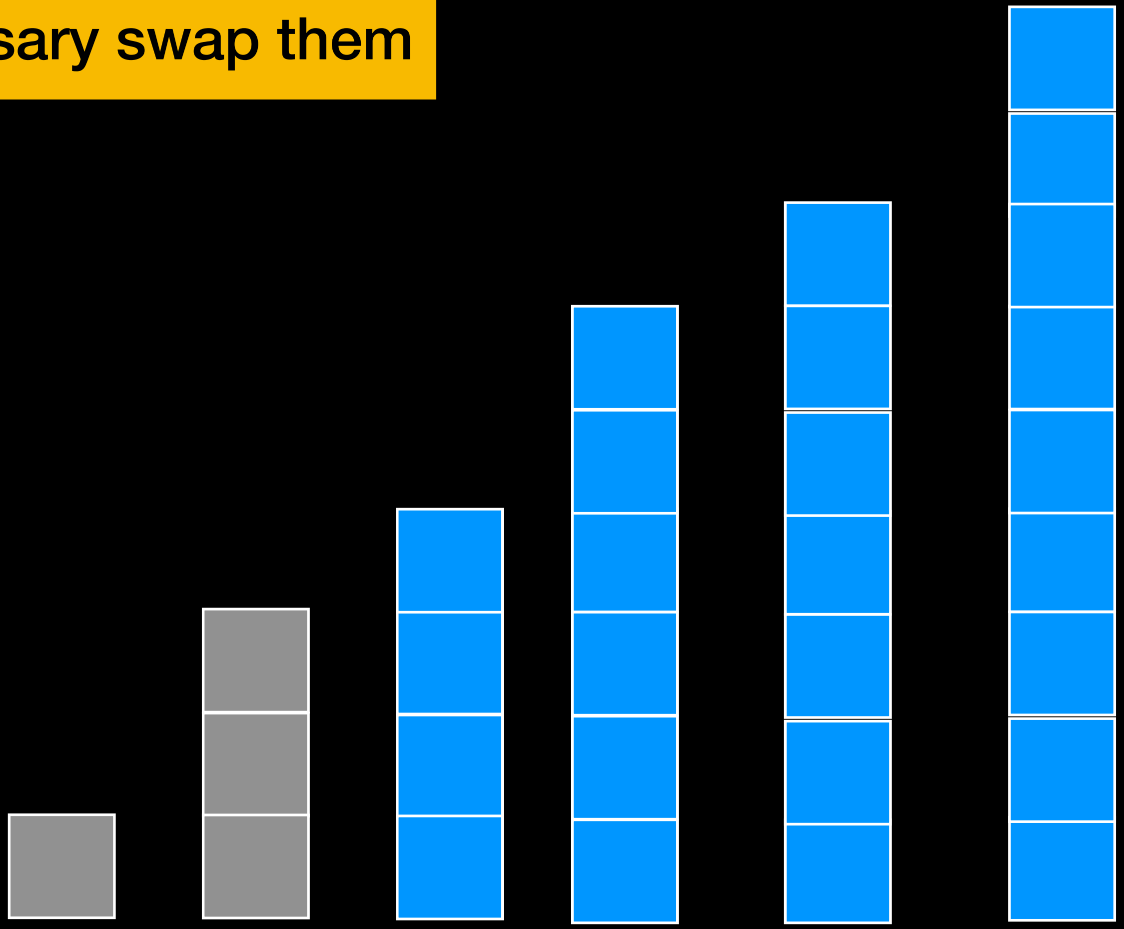
4th Pass



Bubble Sort

Compare adjacent elements and if necessary swap them

5th Pass:
Sort **n-4**



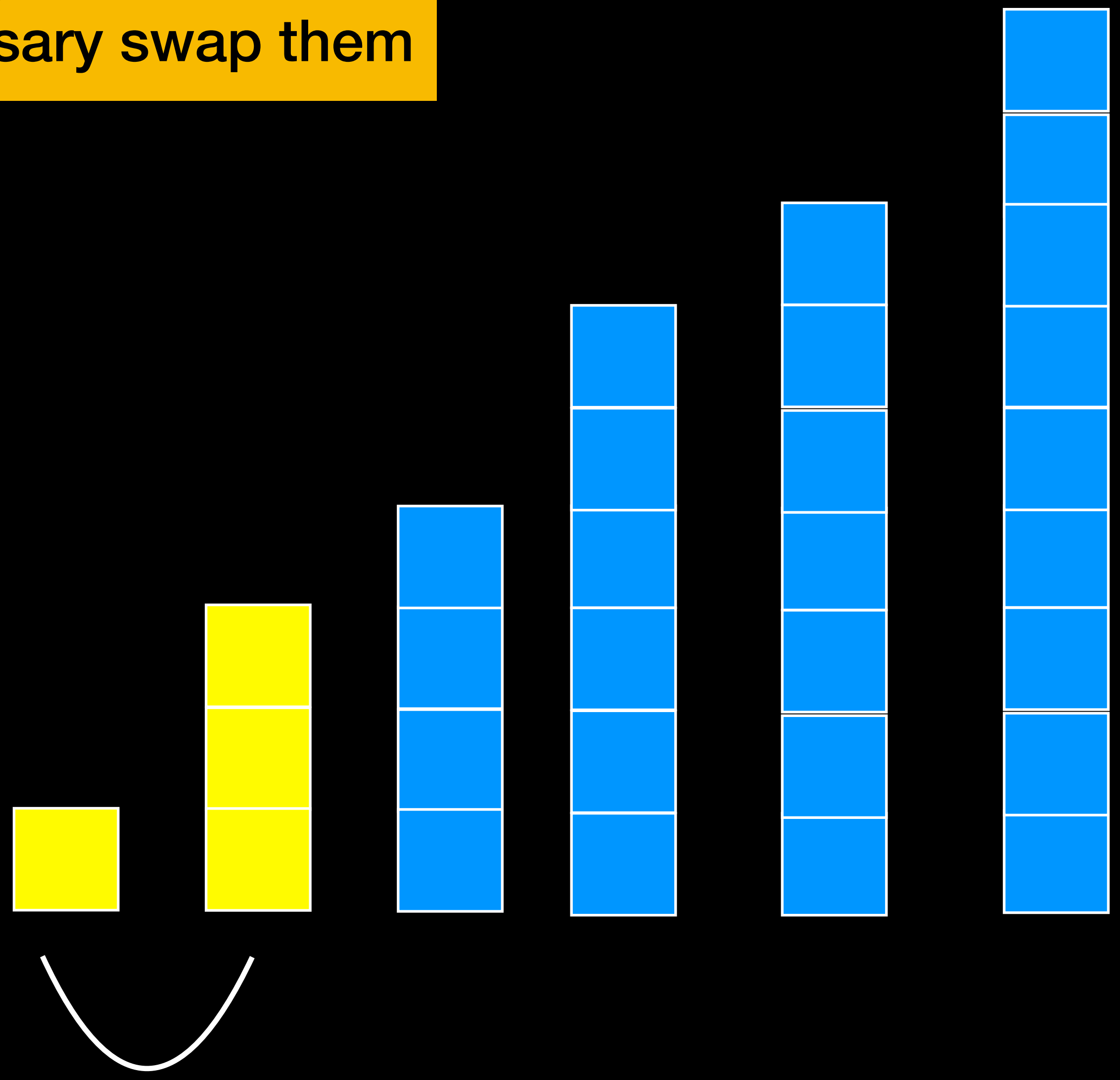
Bubble Sort

⬜ **Unsorted**

🟦 **Sorted**

Compare adjacent elements and if necessary swap them

5th Pass



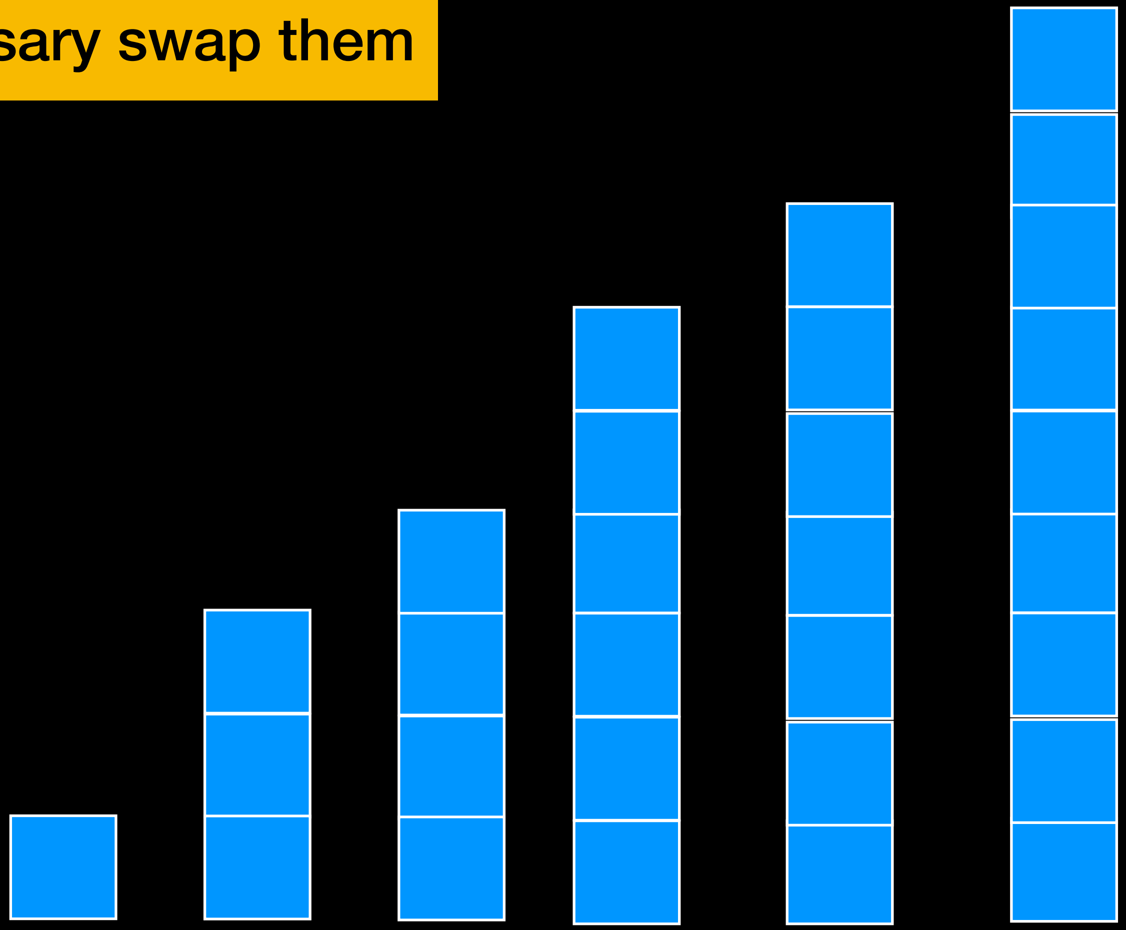
Bubble Sort

▨ Unsorted

🟦 Sorted

Compare adjacent elements and if necessary swap them

Done!



Bubble Sort Analysis

How much work?

First pass: **n-1** comparisons and at most **n-1** swaps

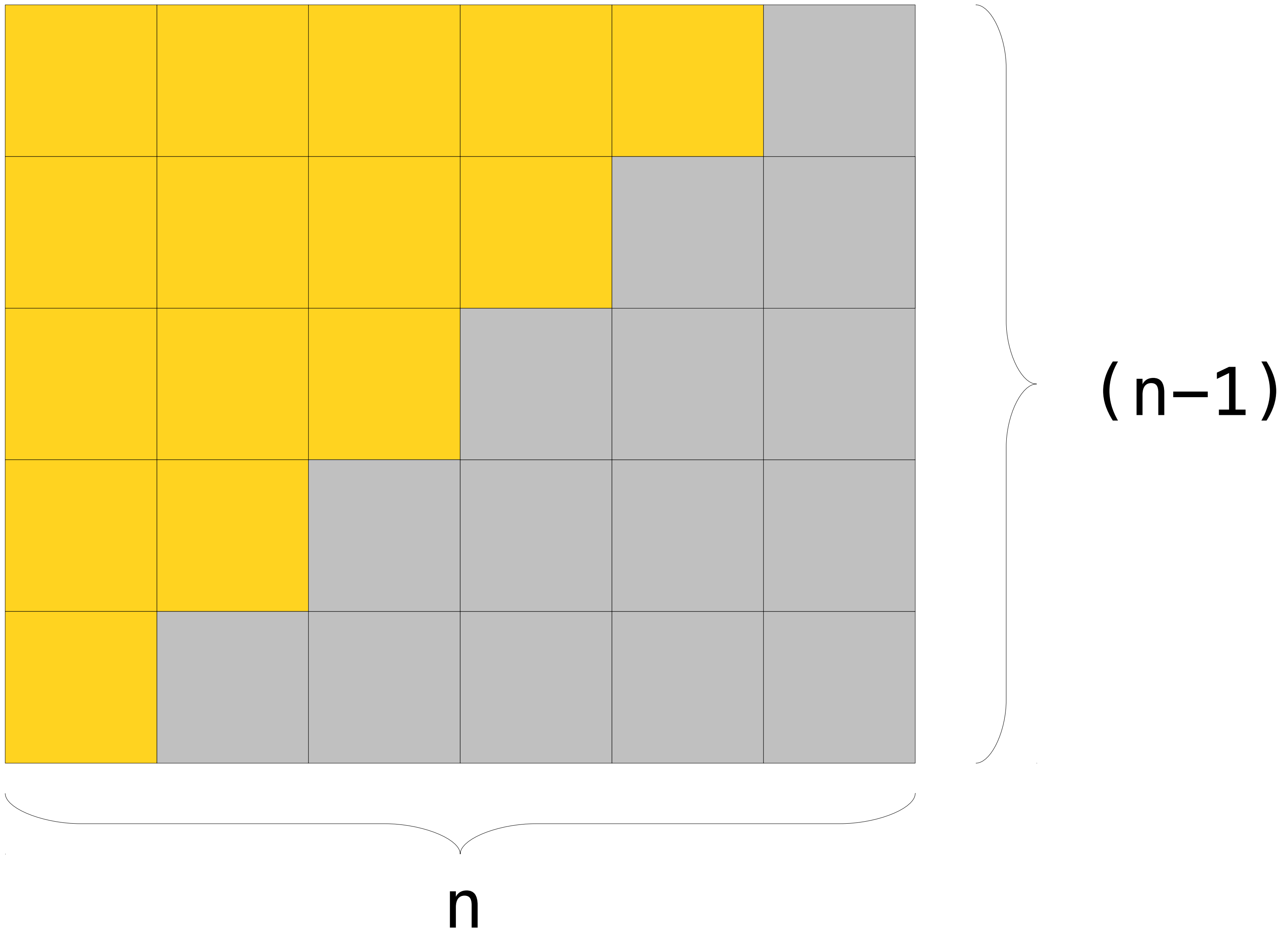
Second pass: **n-2** comparisons and at most **n-2** swaps

Third pass: **n-3** comparisons and at most **n-3** swaps

. . .

Total work: **(n-1) + (n-2) + . . . +1**

$$(n-1) + (n-2) + \ldots + 2 + 1 = n(n-1)/2$$



Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons $+ n(n-1) / 2$ swaps $= O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1) / 2\ ) = O(\ )$?

Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons $+ n(n-1) / 2$ swaps $= O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(n(n-1) / 2)= O(\ )$?

$T(n) = 2((n^2-n) / 2)= O(\ )$?

Bubble Sort Analysis

$T(n)$ = n(n-1) / 2 comparisons + n(n-1) / 2 swaps = $O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1) / 2\ ) = O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ ) = O(\ )$?

$T(n) = n^2-n = O(\ )$?

Ignore non-dominant terms

Bubble Sort Analysis

$T(n)$ = n(n-1) / 2 comparisons + n(n-1) / 2 swaps = $O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1)\ /\ 2\ ) = O(\ )$?

$T(n) = 2(\ (n^2-n)\ /\ 2\ ) = O(\ )$?

$T(n) = n^2-n = O(\ )$?

Bubble Sort run time is **$O(n^2)$**

Optimize!

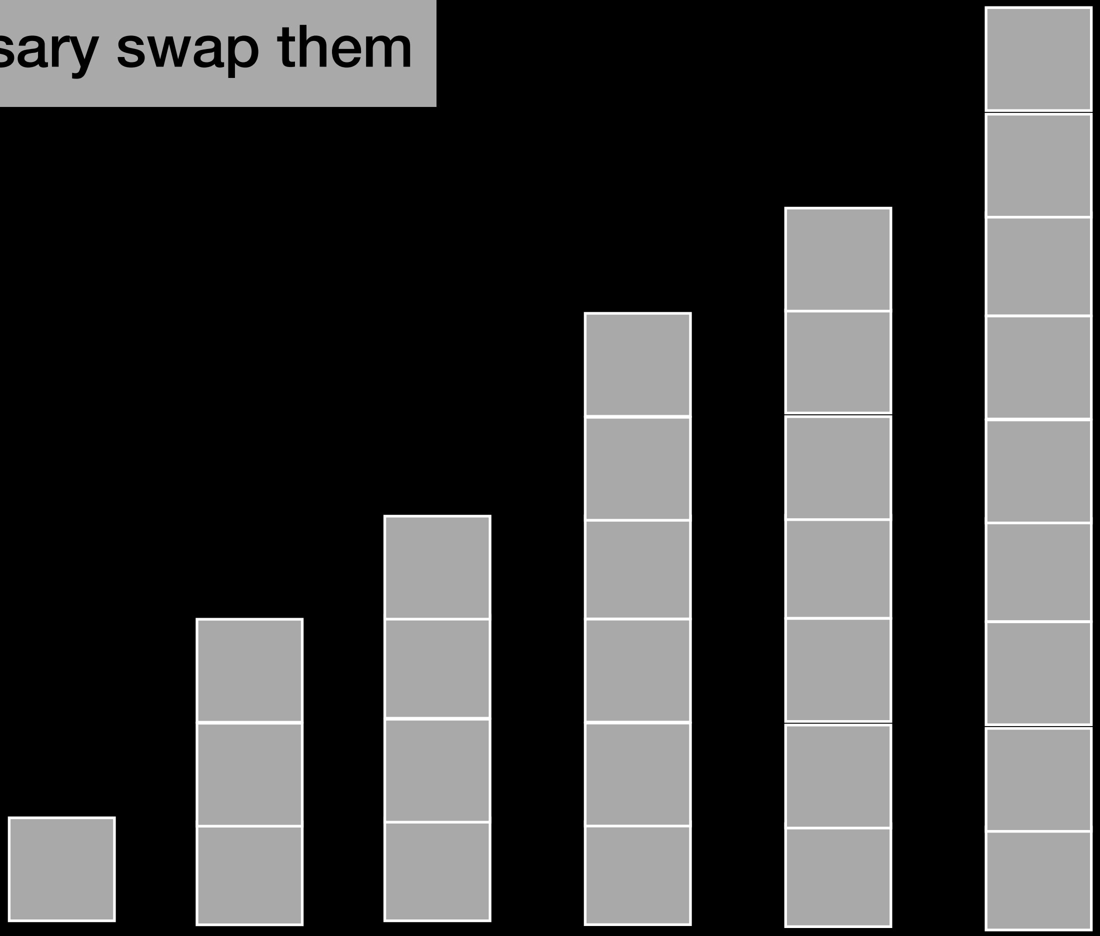
Easy to check:

if there are no swaps in any given pass

stop because it is sorted

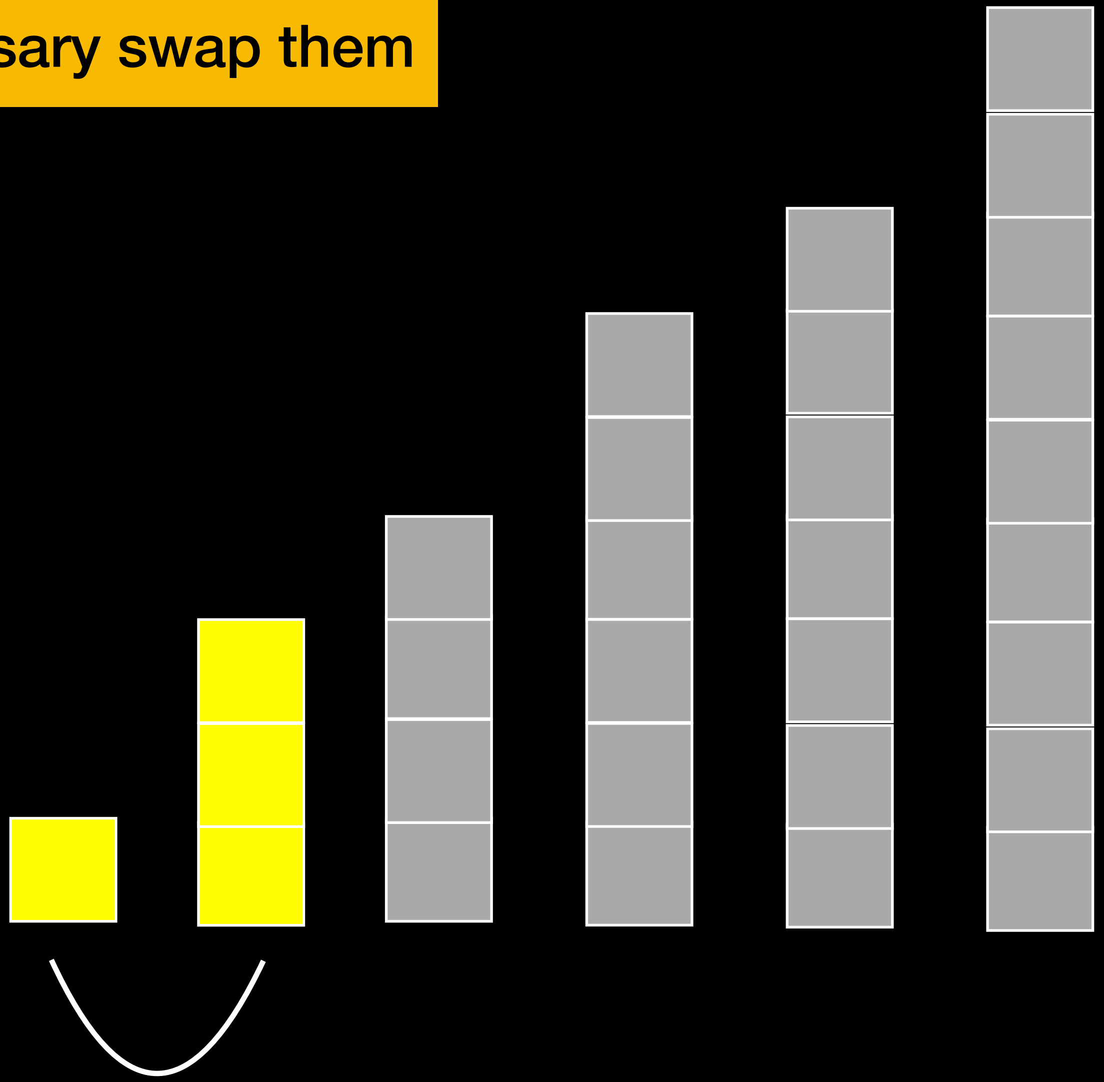
Bubble Sort

Compare adjacent elements and if necessary swap them



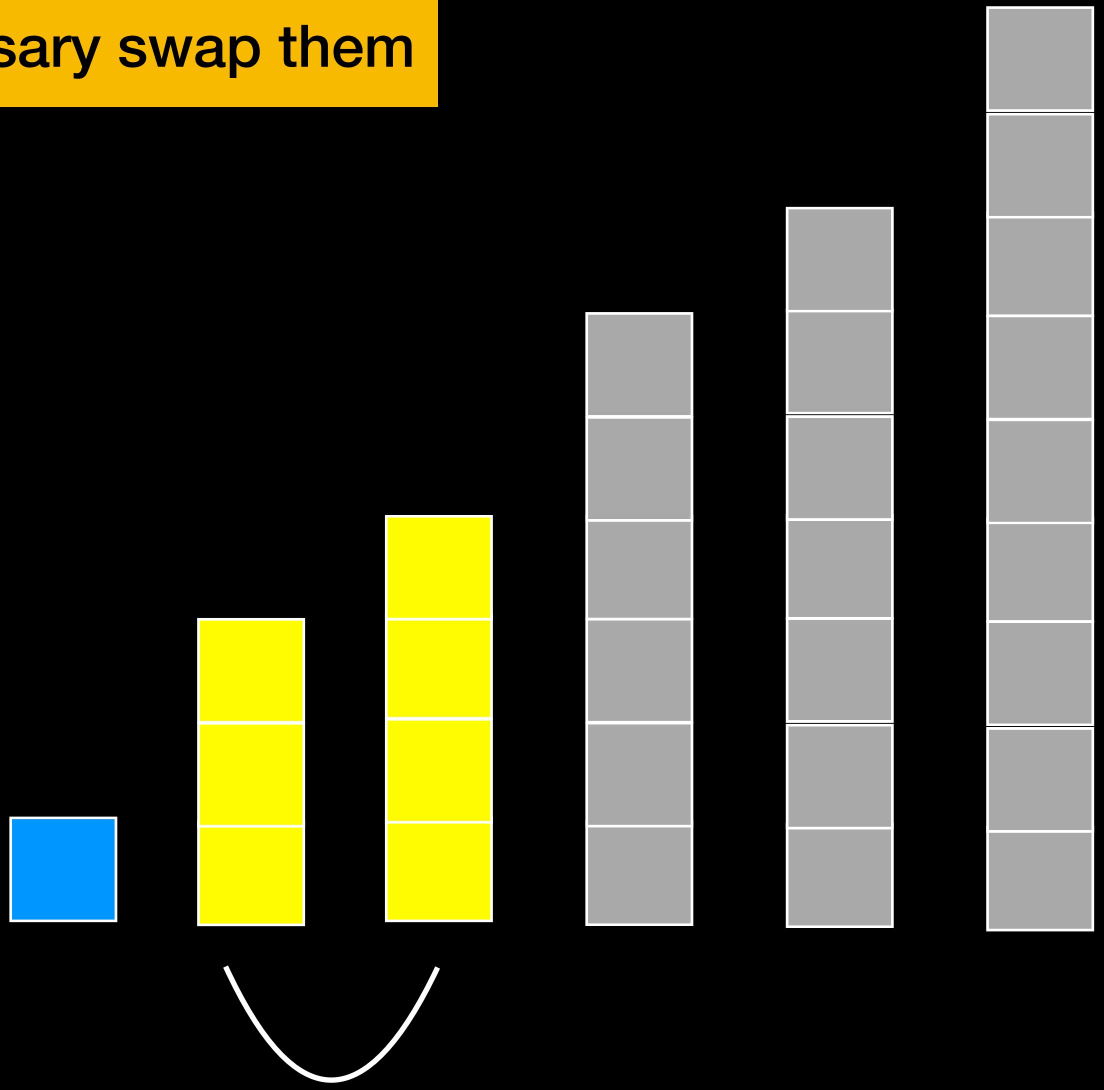
Bubble Sort

Compare adjacent elements and if necessary swap them



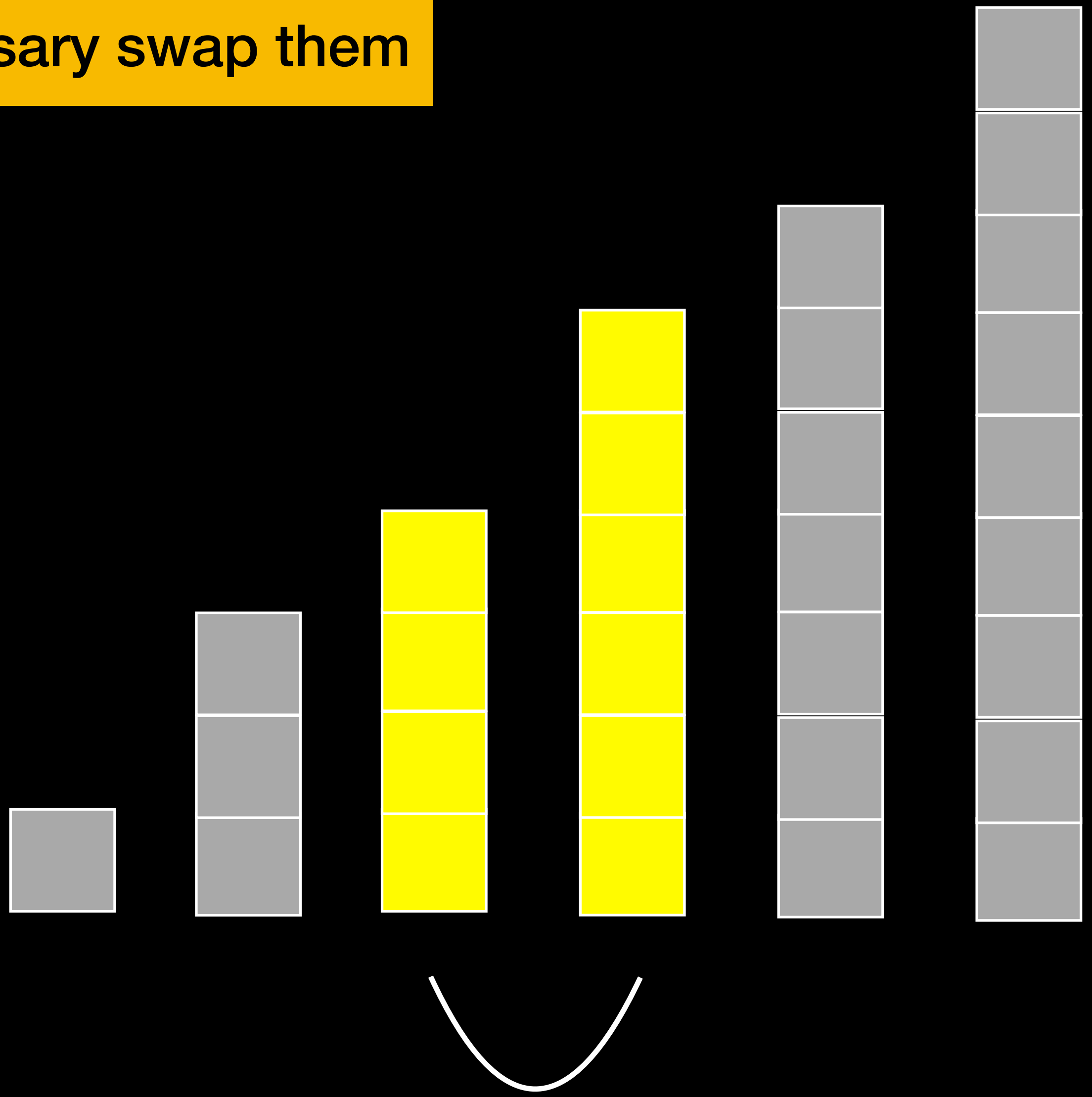
Bubble Sort

Compare adjacent elements and if necessary swap them



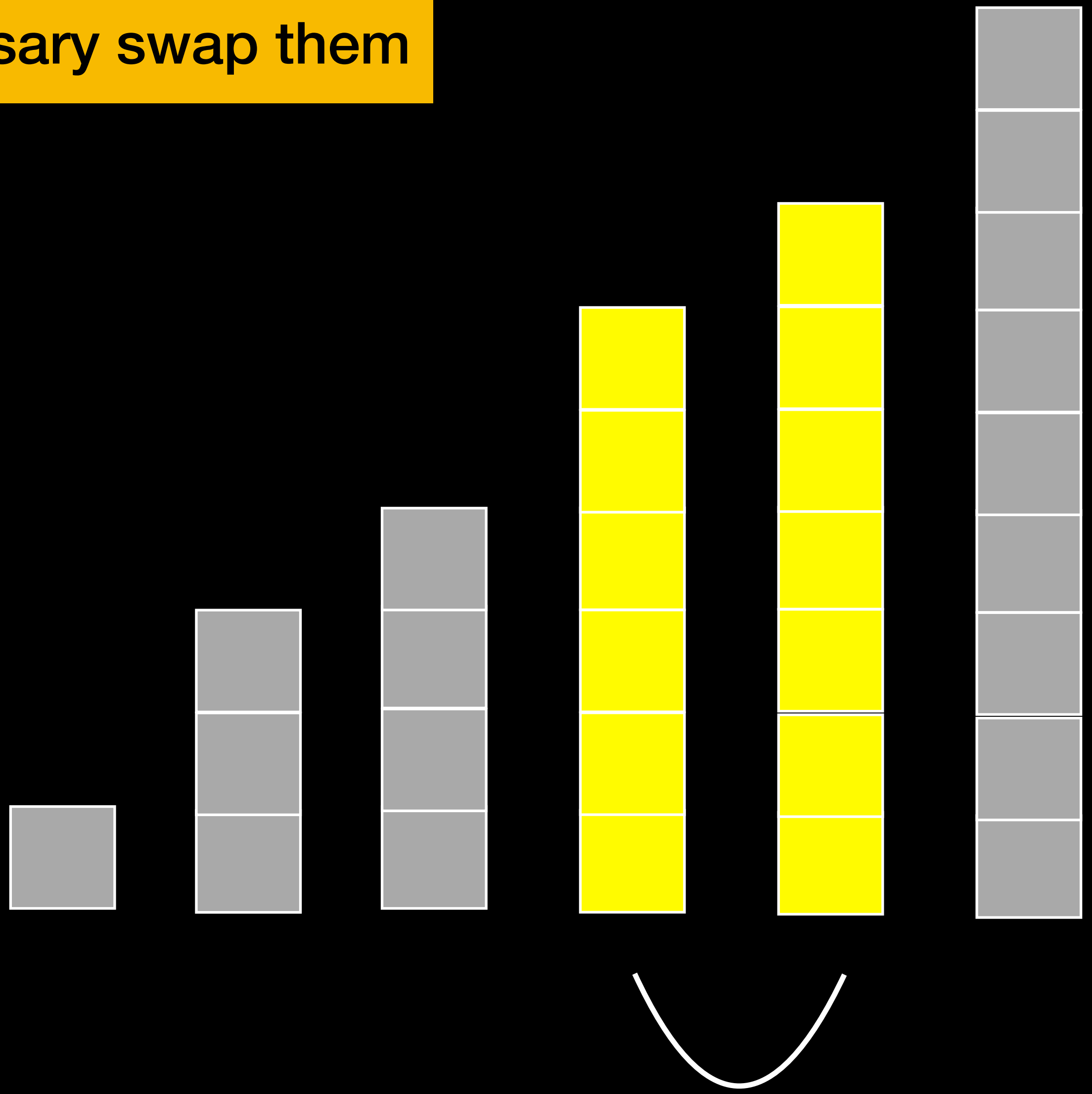
Bubble Sort

Compare adjacent elements and if necessary swap them



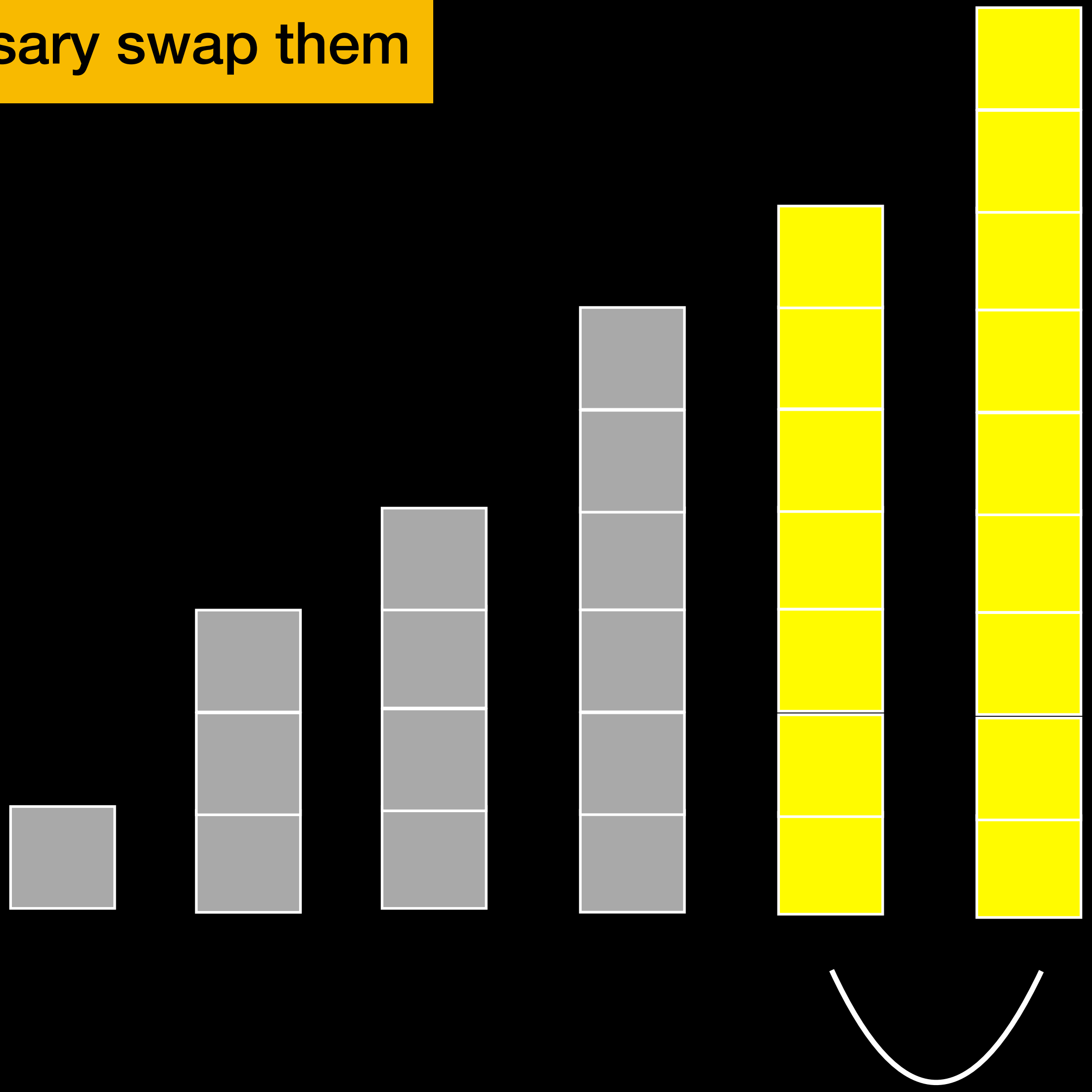
Bubble Sort

Compare adjacent elements and if necessary swap them



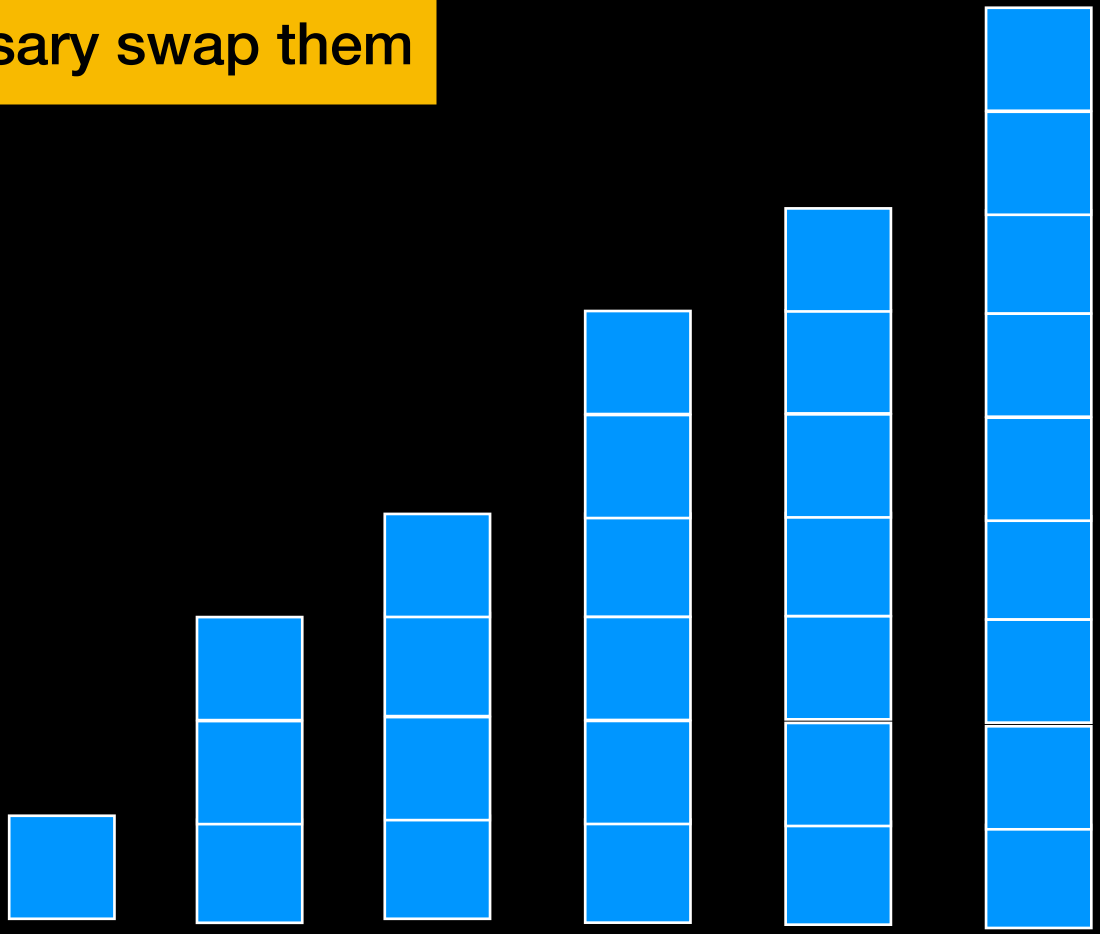
Bubble Sort

Compare adjacent elements and if necessary swap them



Bubble Sort

Compare adjacent elements and if necessary swap them



```
template<typename ItemType>
void bubbleSort(ItemType the_array[], size_t size)
{
   bool swapped = true; // Assume unsorted
   int pass = 1;
   while (swapped && (pass < size))
   {
      // At this point, if pass > 1 the_array[size+1-pass … size-1]
      // is sorted and all of its entries are > the entries in
      // the_array[0 ... size-pass]
      swapped = false;
      for (int index = 0; index < size - pass; index++)
      {
         // At this point, all entries in the_array[0 ... index-1]
         // are <= the_array[index]
         if (the_array[index] > the_array[index+1])
         {
             swap(the_array[index], the_array[index+1]); //swap
             swapped = true; // indicates array not yet sorted
         }// end if
      }  // end for
      //Assertion:the_array[0 ... size-pass-1]<the_array[size-pass]
      pass++;
   }  // end while
}  // end bubbleSort
```

```
template<typename ItemType>
void bubbleSort(ItemType the_array[], size_t size)
{
   bool swapped = true; // Assume unsorted
   int pass = 1;
Pass while (swapped && (pass < size))
O(n){
      // At this point, if pass > 1 the_array[size+1-pass … size-1]
      // is sorted and all of its entries are > the entries in
      // the_array[0 ... size-pass]
       swapped = false;
  O(n)for (int index = 0; index < size - pass; index++)
      {
         // At this point, all entries in the_array[0 ... index-1]
         // are <= the_array[index]
         if (the_array[index] > the_array[index+1])
         {
             swap(the_array[index], the_array[index+1]); //swap
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         }// end if
      }  // end for
      //Assertion:the_array[0 ... size-pass-1]<the_array[size-pass]
      pass++;
   }  // end while
}  // end bubbleSort
```

$O(\ n^2)$

Bubble Sort Analysis

Execution time DOES depend on initial arrangement of data

Worst case: O(n^2) comparisons and data moves

Best case: O(n) comparisons and data moves

Stable

If array is already sorted bubble sort will stop after first pass and no swaps => good choice for small n and data likely somewhat sorted

Raise your hand if you had Bubble Sort

https://www.youtube.com/watch?v=lyZQPjUT5B4

Insertion Sort

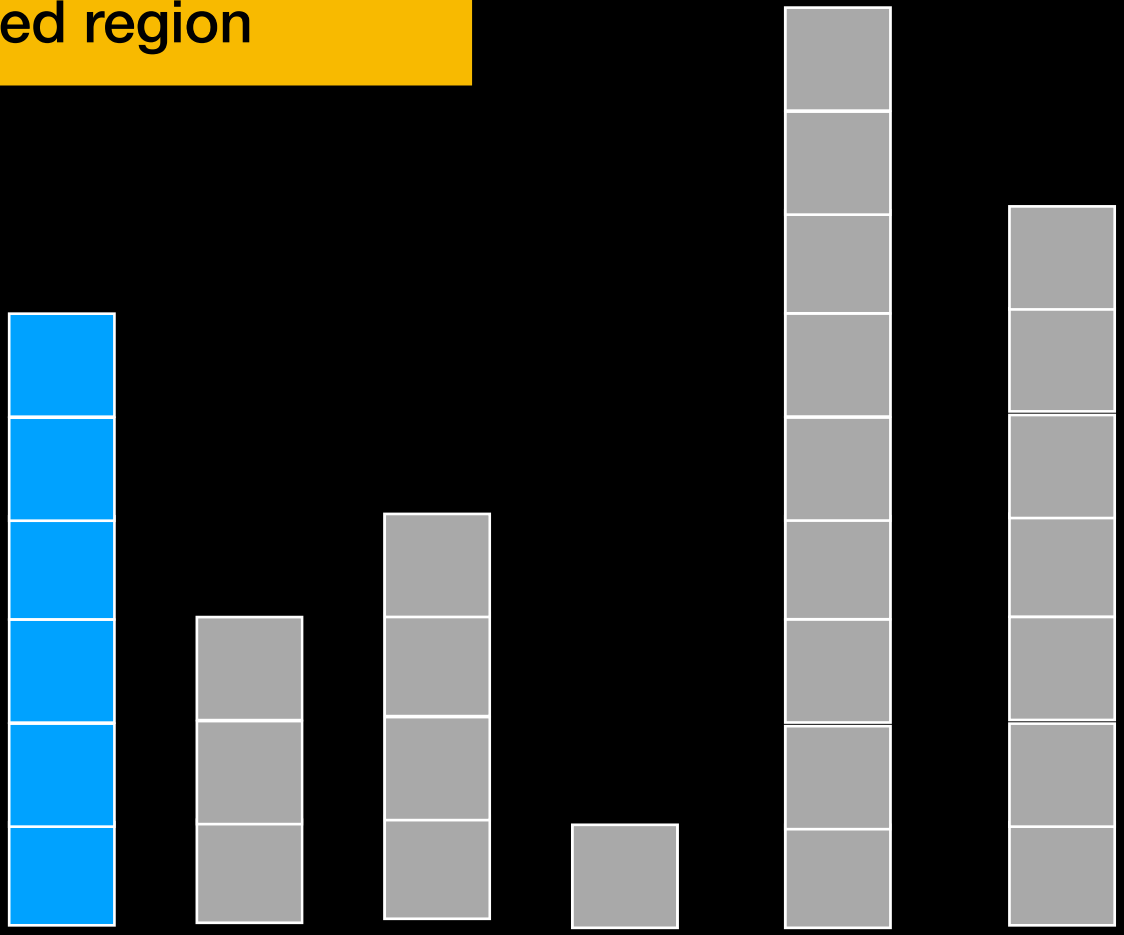
Insertion Sort

◼ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

1st Pass



Insertion Sort

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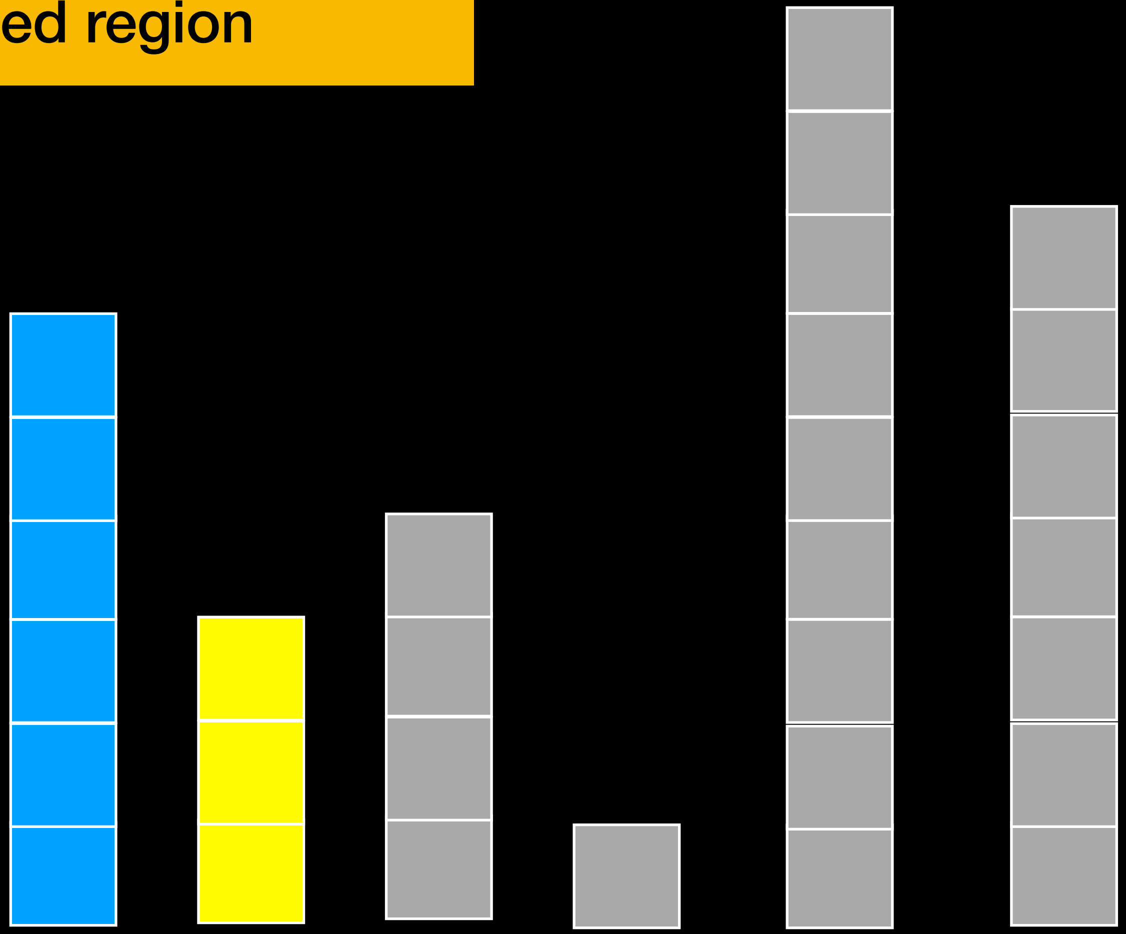
1st Pass



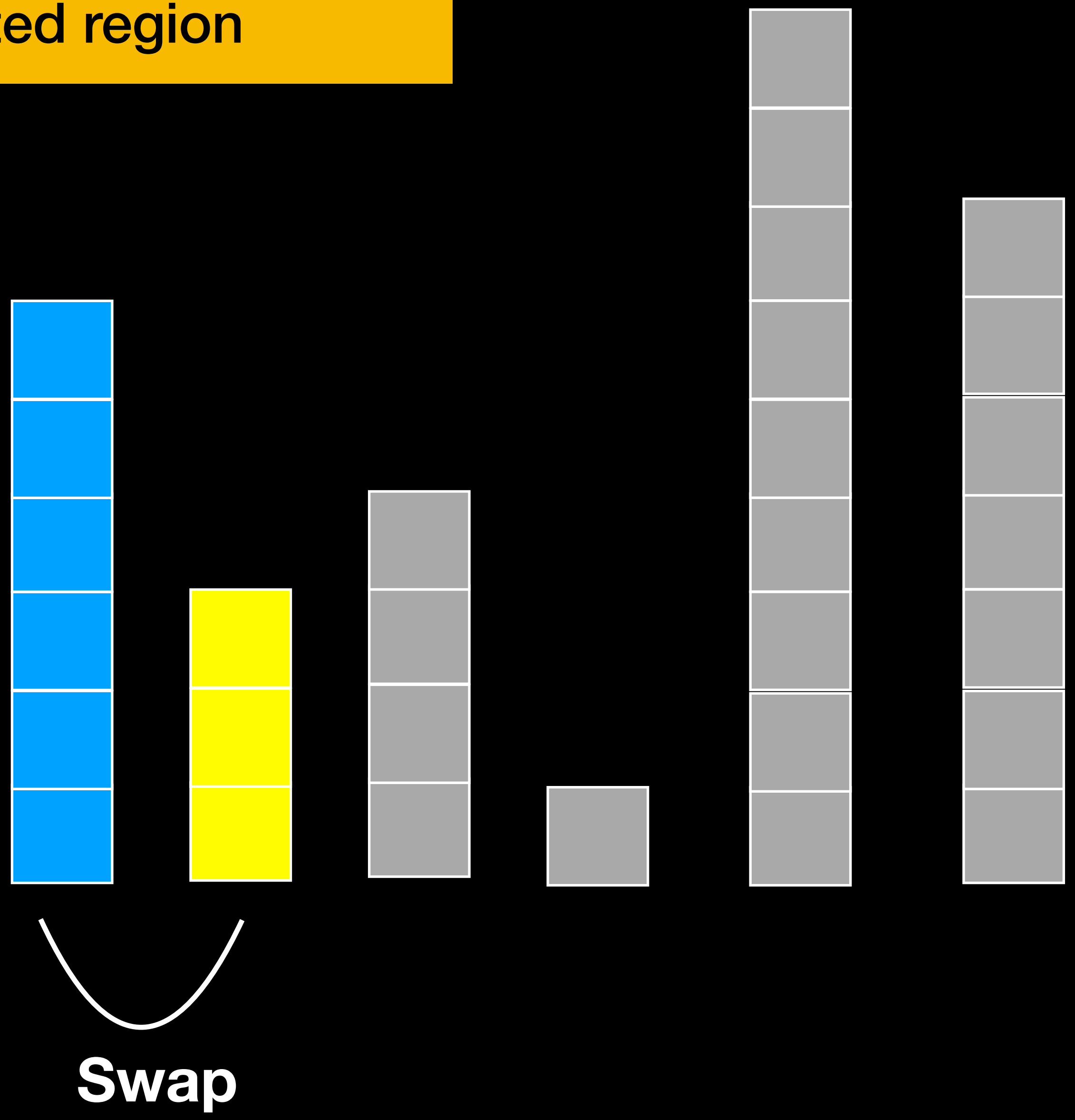
Insertion Sort

▨ Unsorted

🟦 Sorted

Pick first element in unsorted region and put it in right place in sorted region

1st Pass



Insertion Sort

⬜ **Unsorted**

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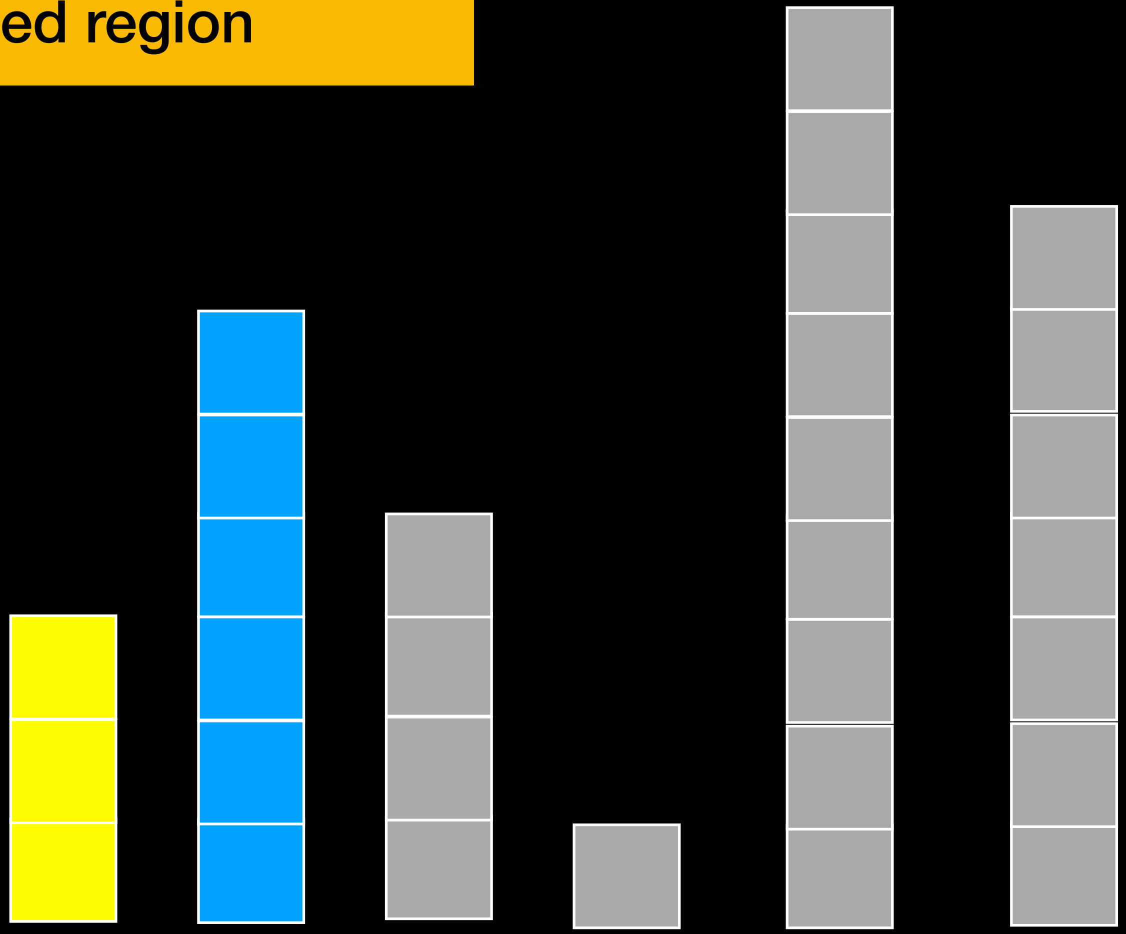
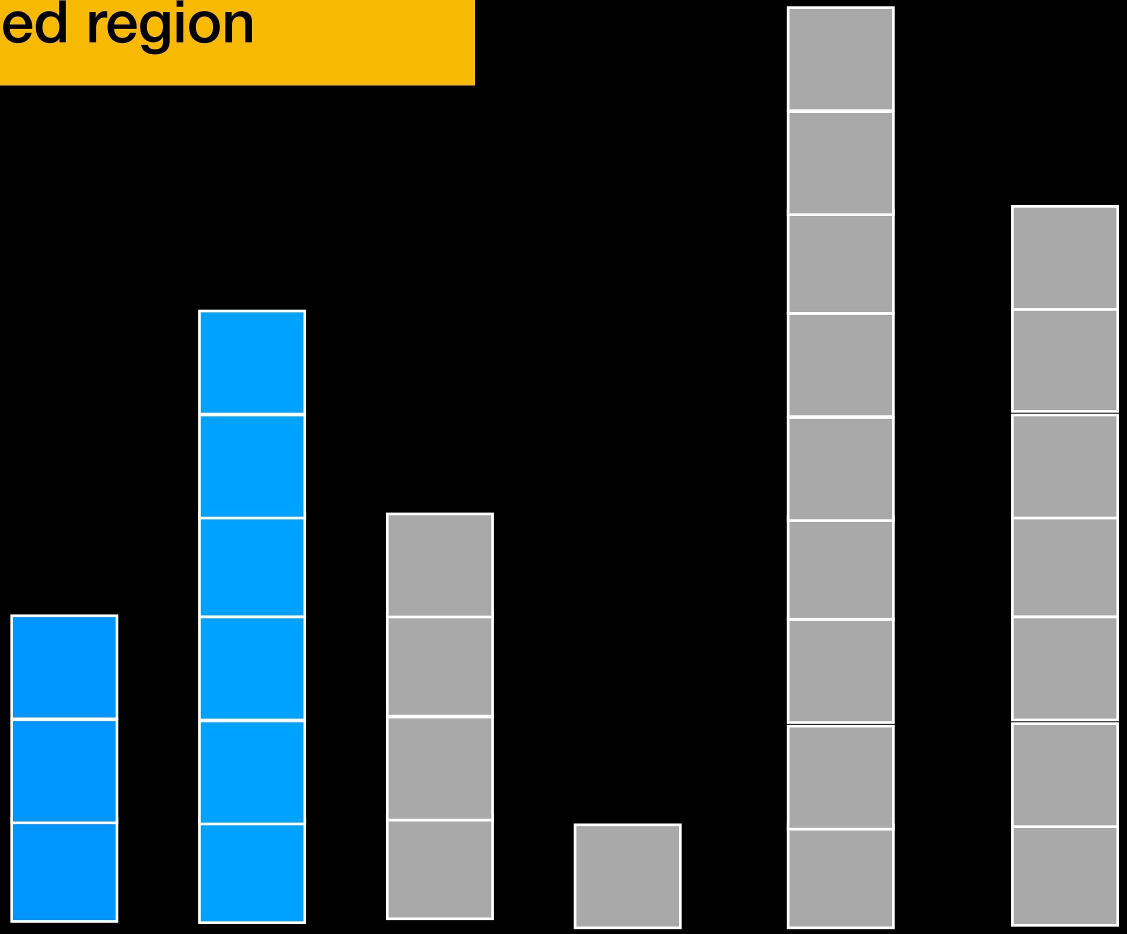
1st Pass



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region



▨ **Unsorted**

🟦 **Sorted**

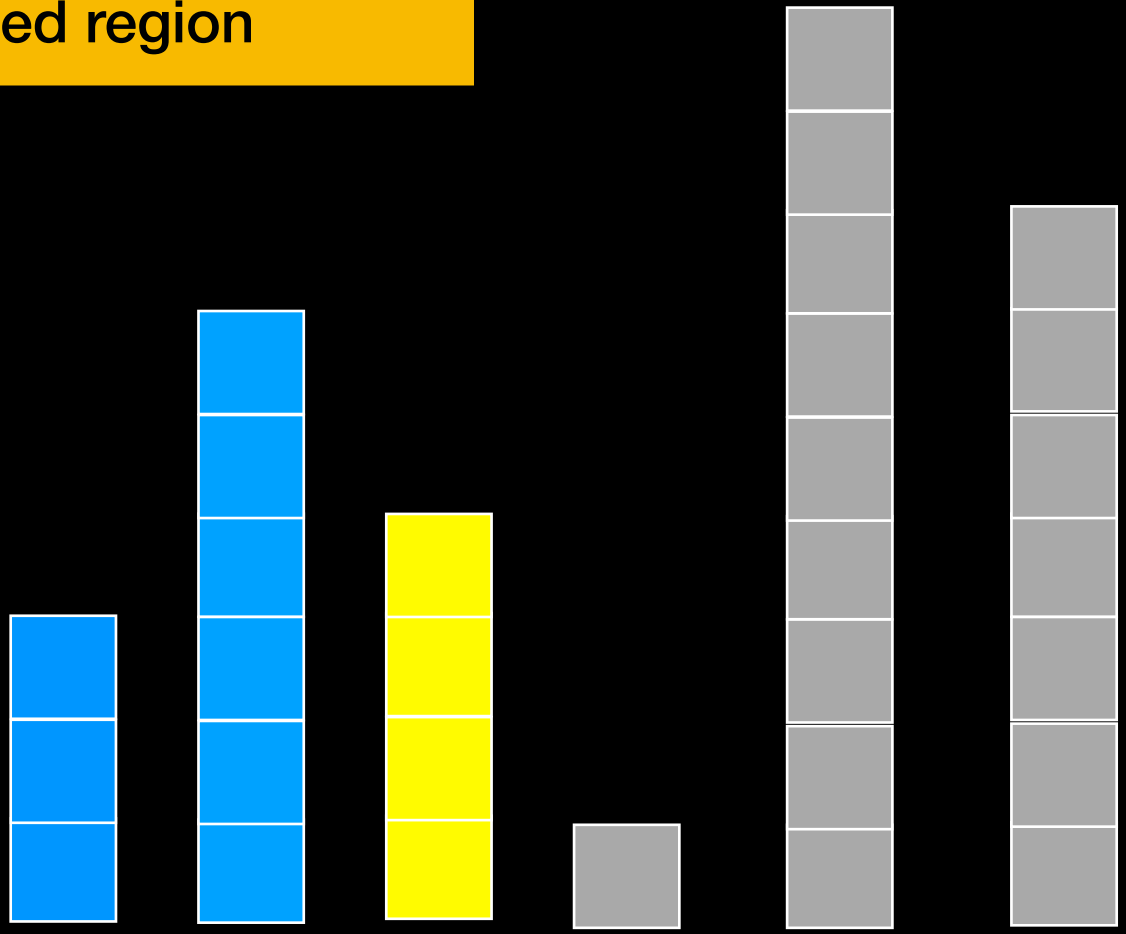
Insertion Sort

⬜ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

2nd Pass



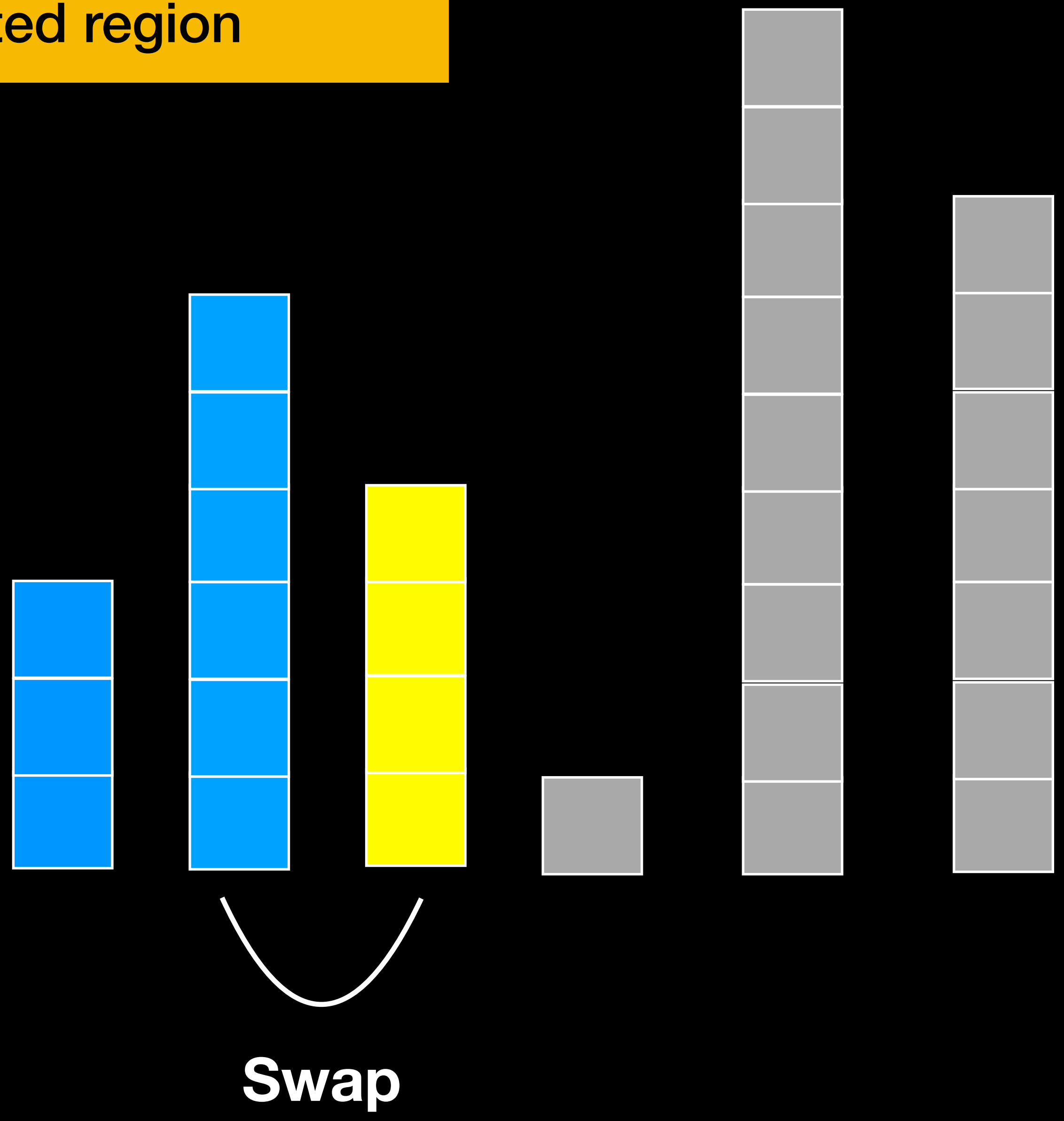
Insertion Sort

⬜ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

2nd Pass



Insertion Sort

⬜ **Unsorted**

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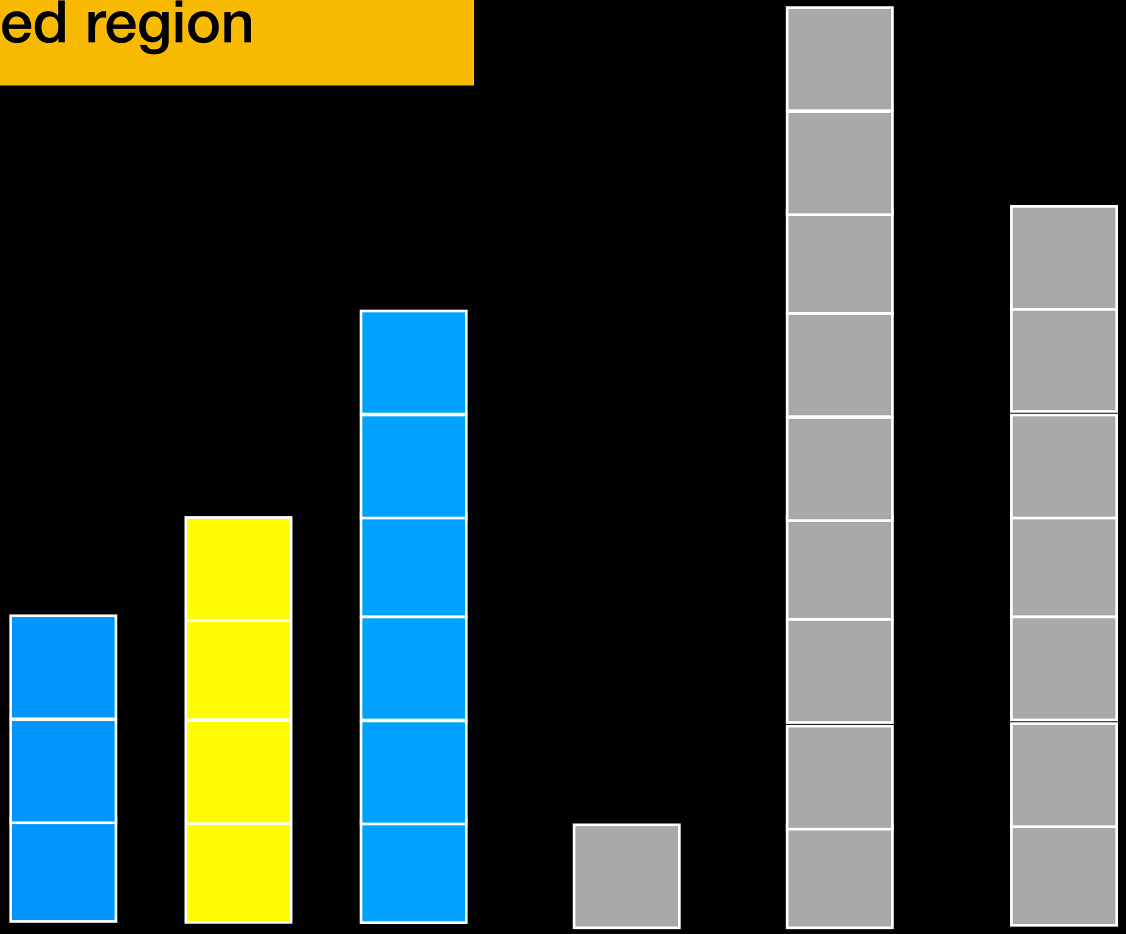
Pick first element in unsorted region and put it in right place in sorted region

2nd Pass



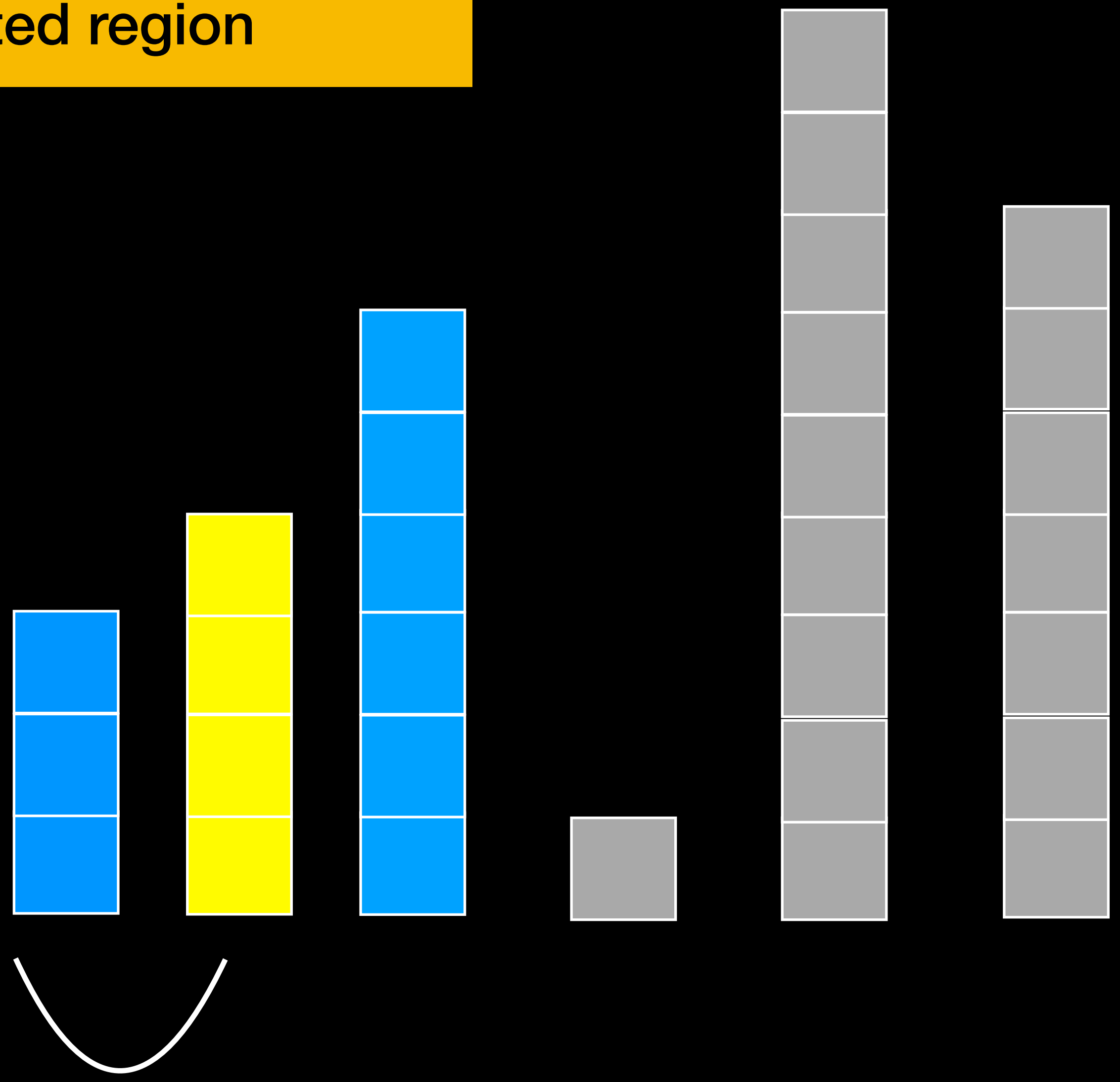
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

🟦 **Sorted**

2nd Pass

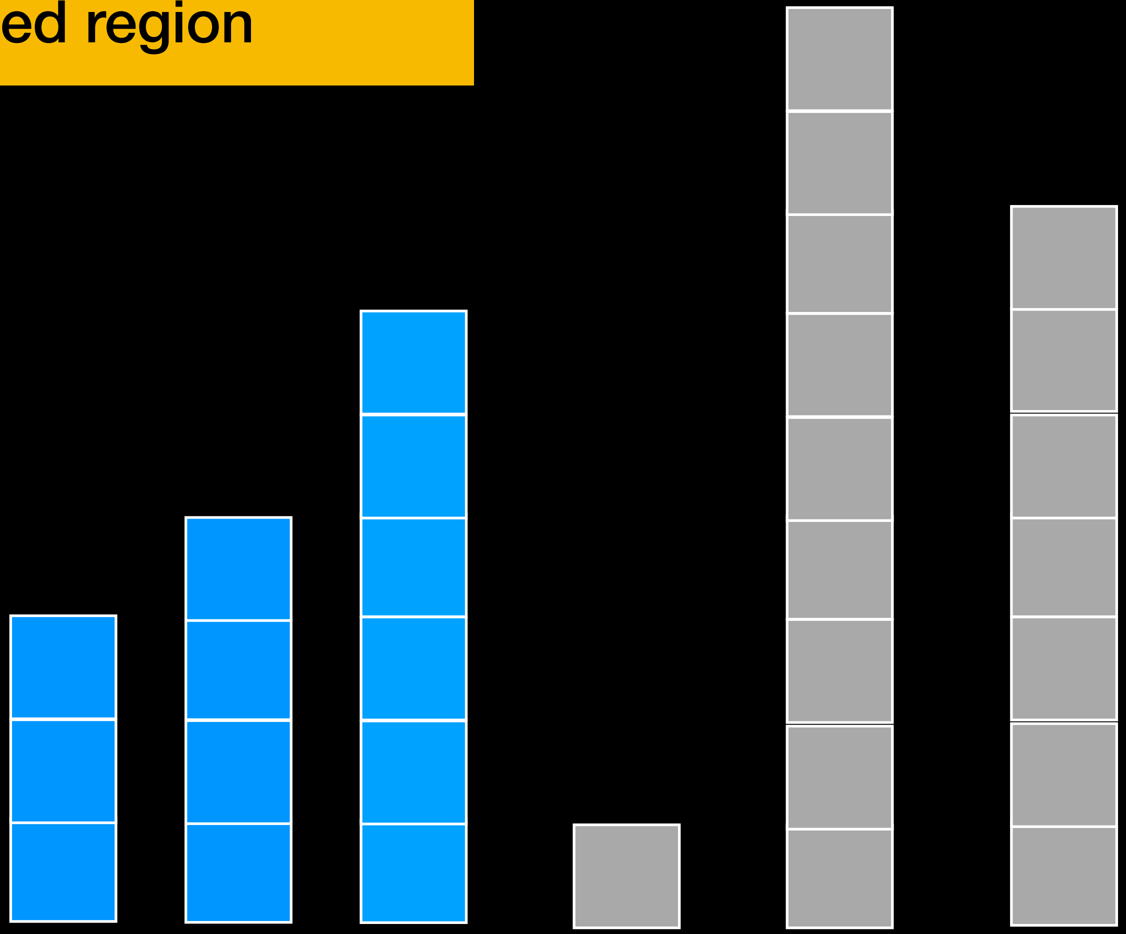


Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region



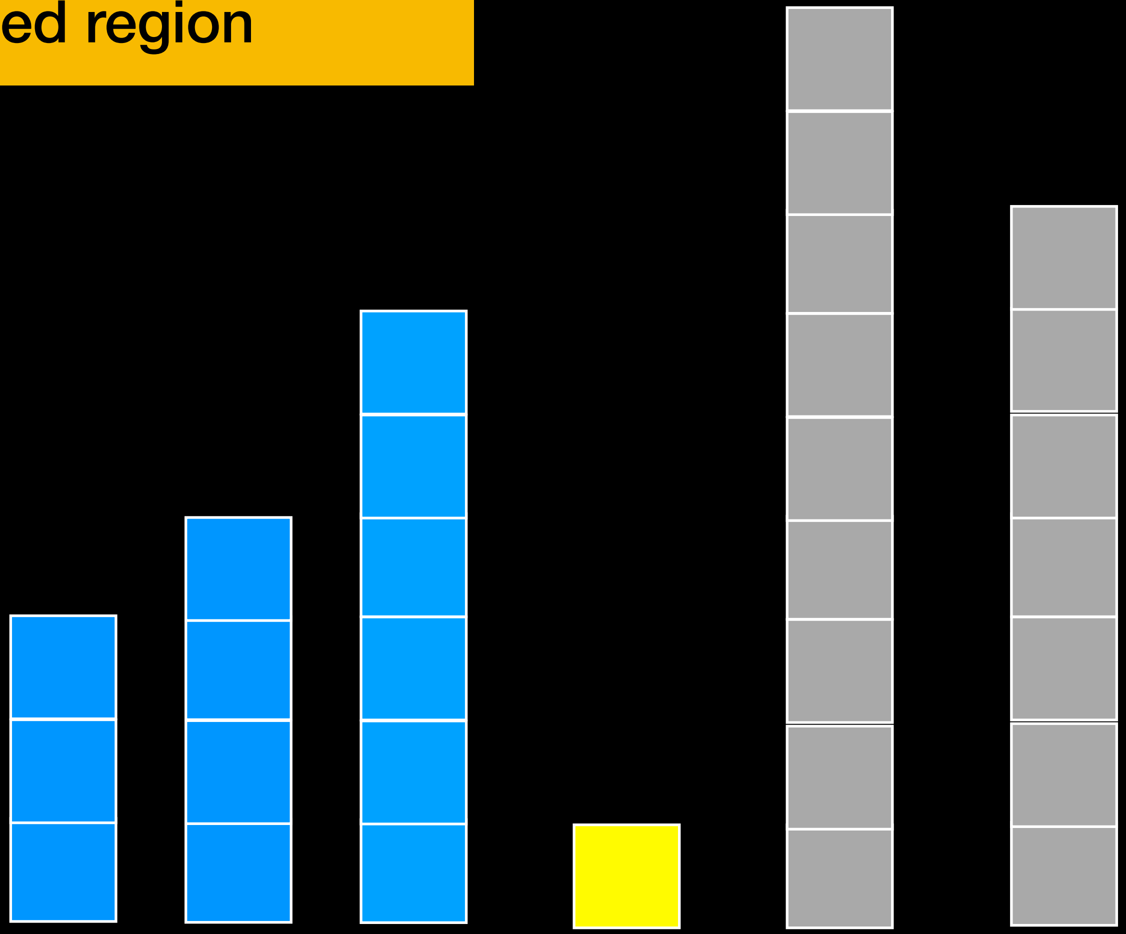
Insertion Sort

⬜ Unsorted

🟦 Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



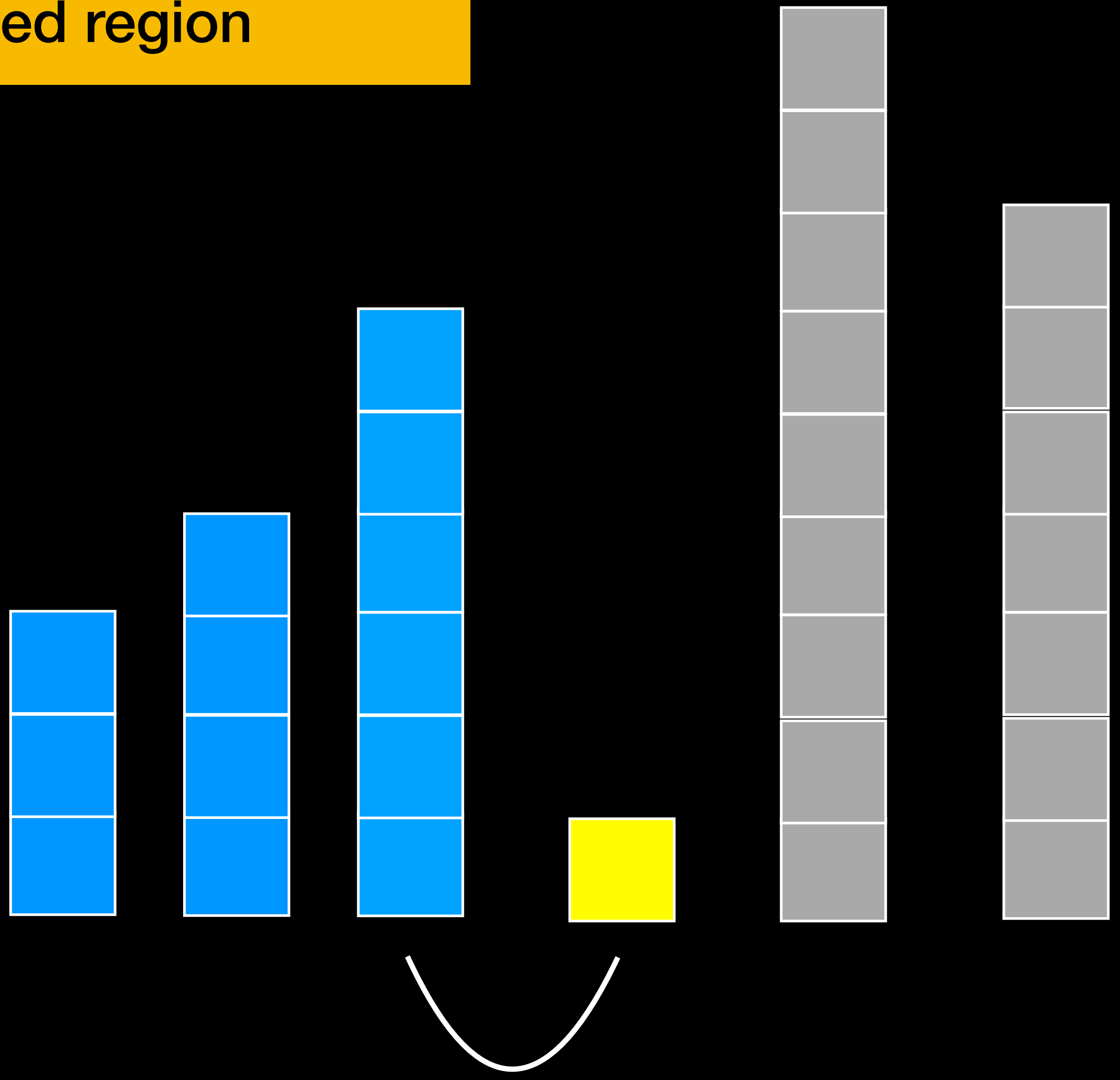
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

🟦 **Sorted**

3rd Pass



Swap

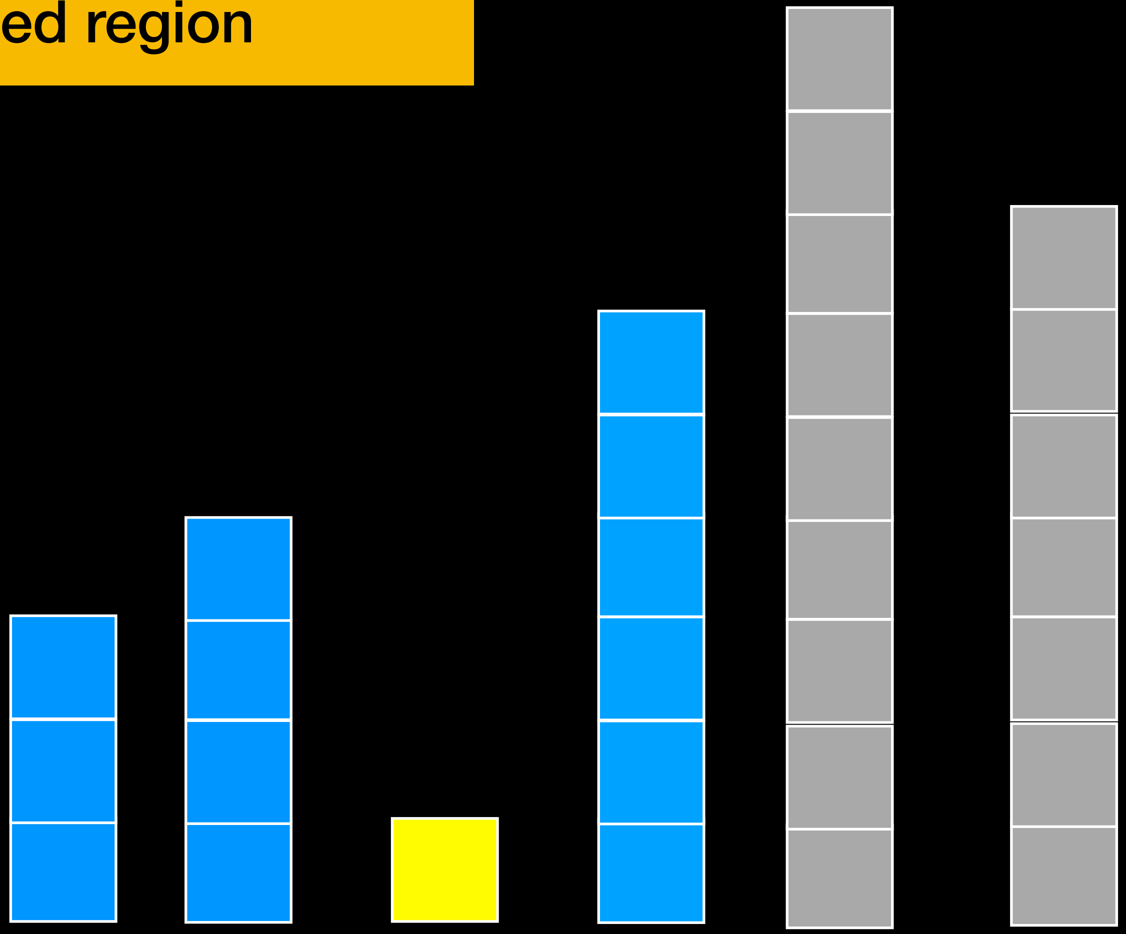
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

🟦 **Sorted**

3rd Pass



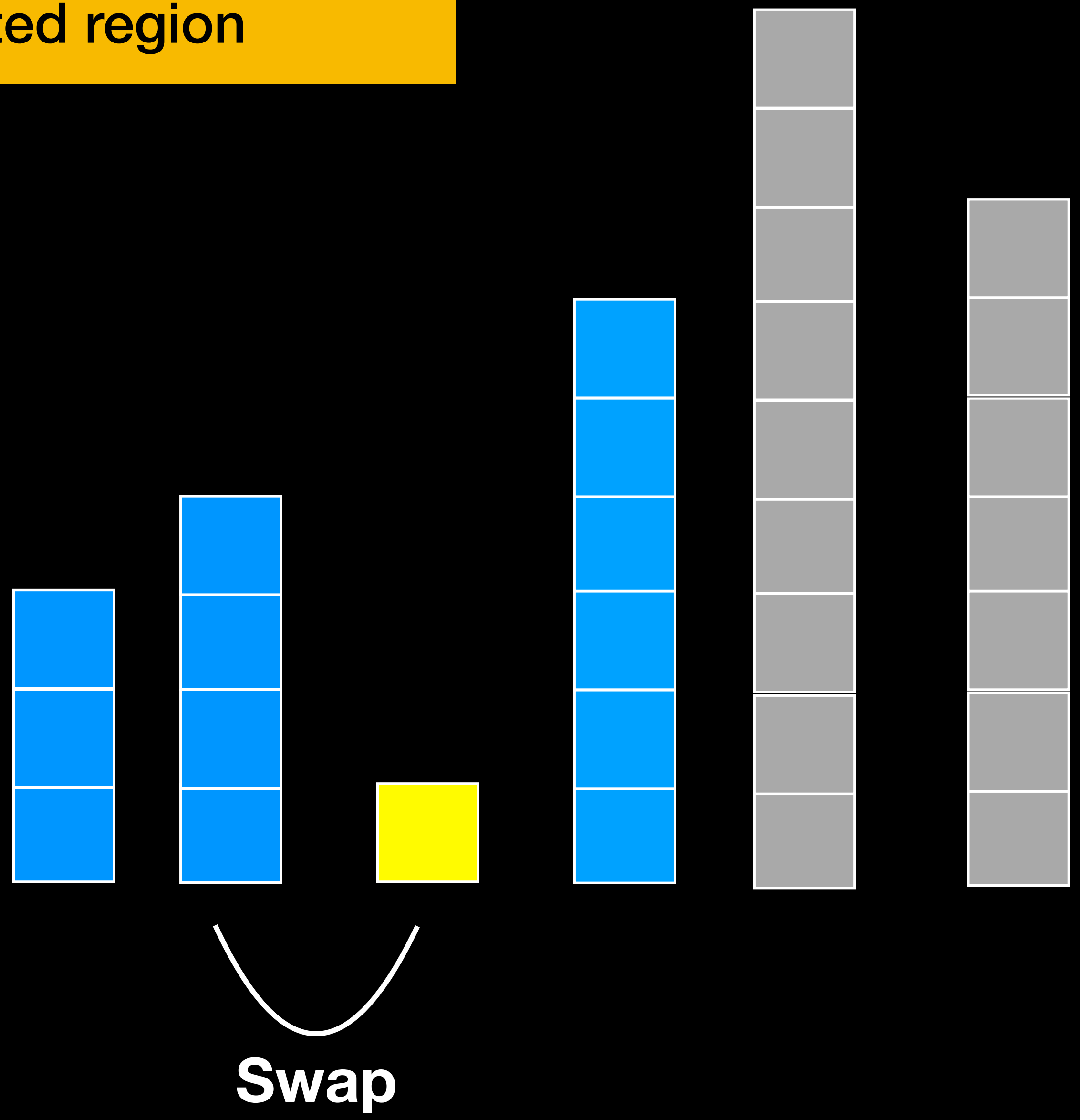
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ Unsorted

🟦 Sorted

3rd Pass



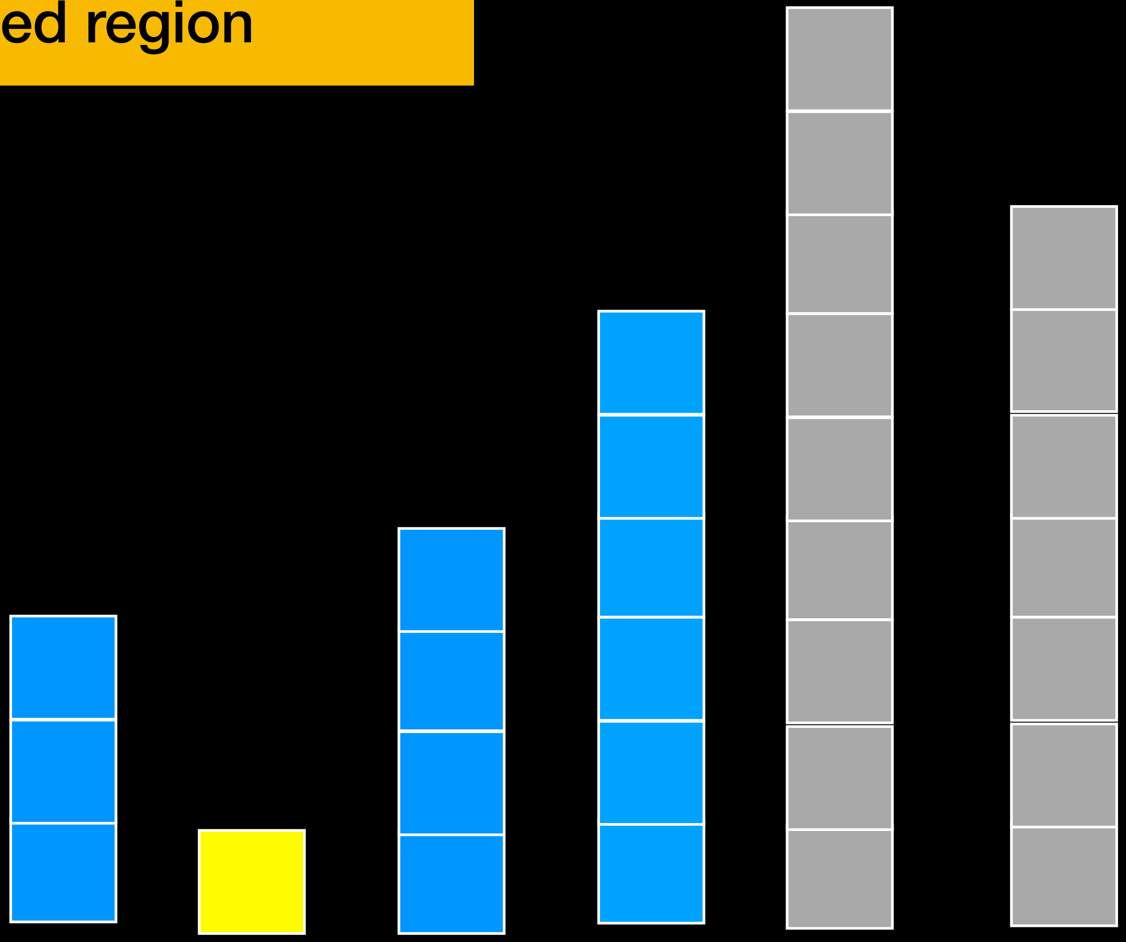
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

🟦 **Sorted**

3rd Pass



Insertion Sort

▨ **Unsorted**

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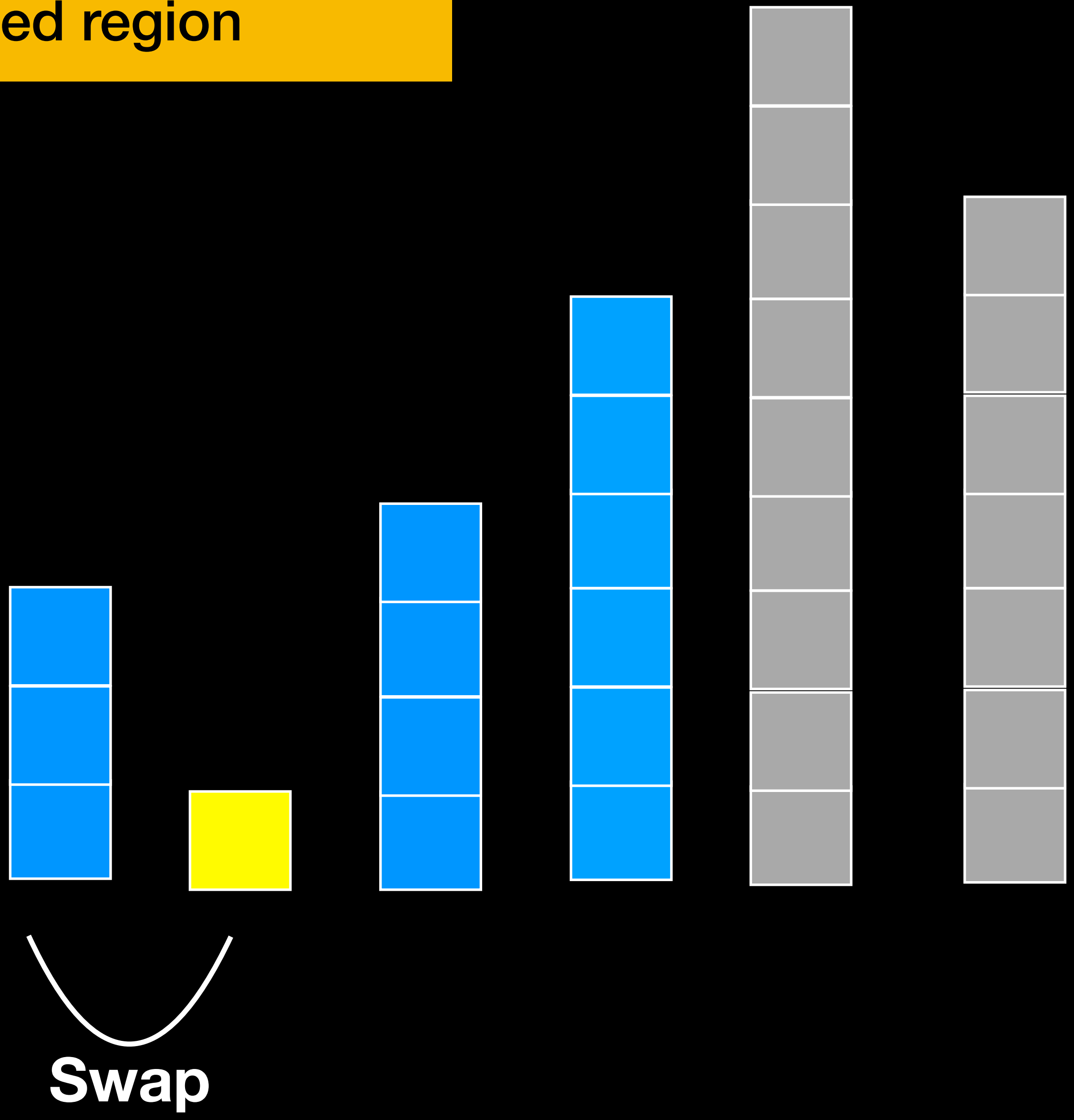
3rd Pass



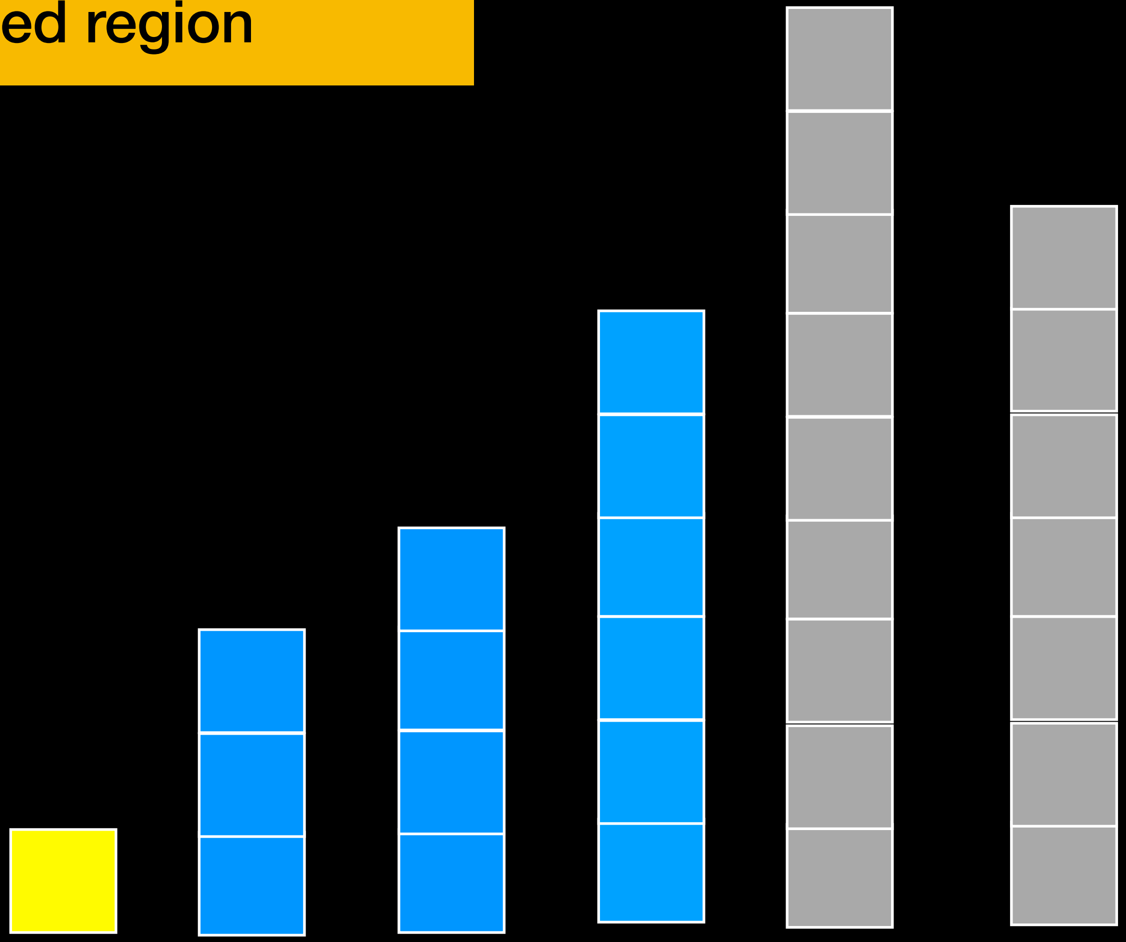
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

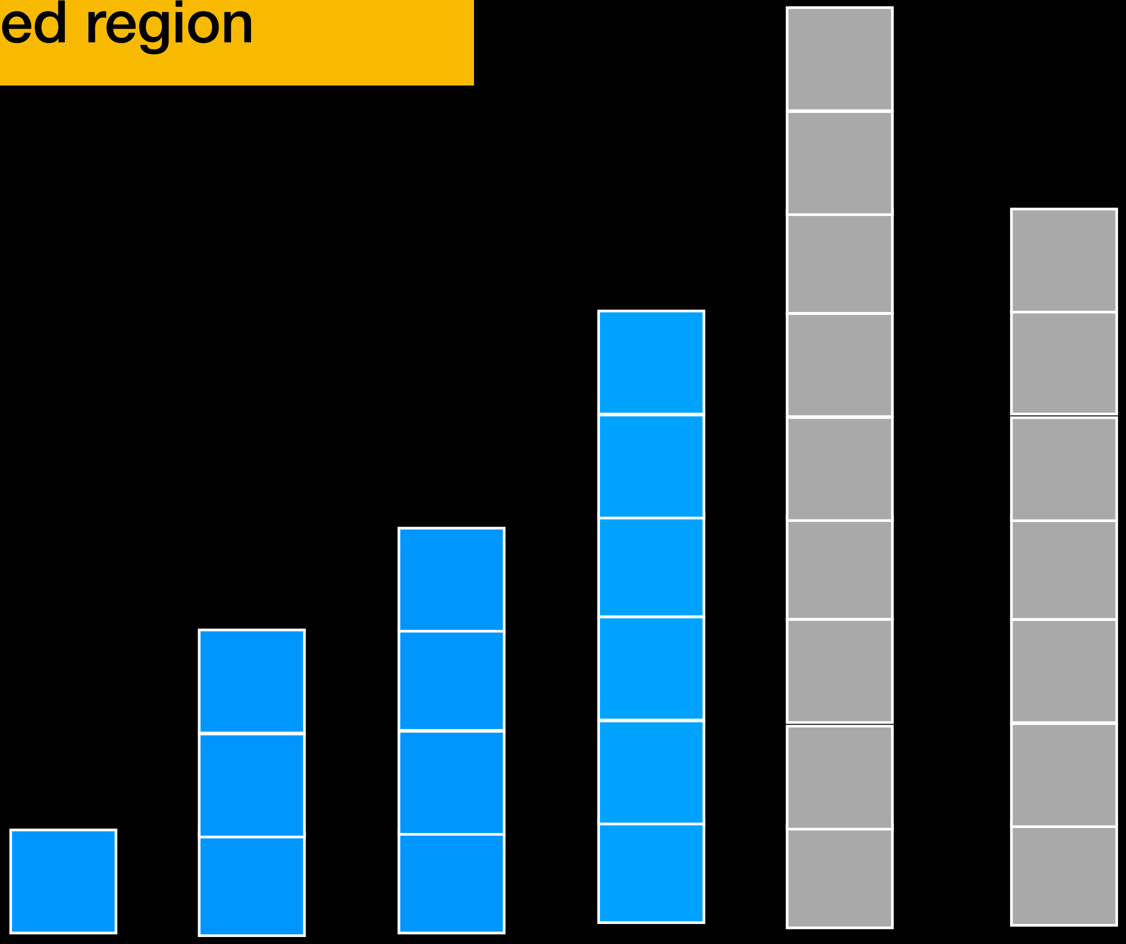
🟦 **Sorted**

3rd Pass



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region



▨ **Unsorted**

🟦 **Sorted**

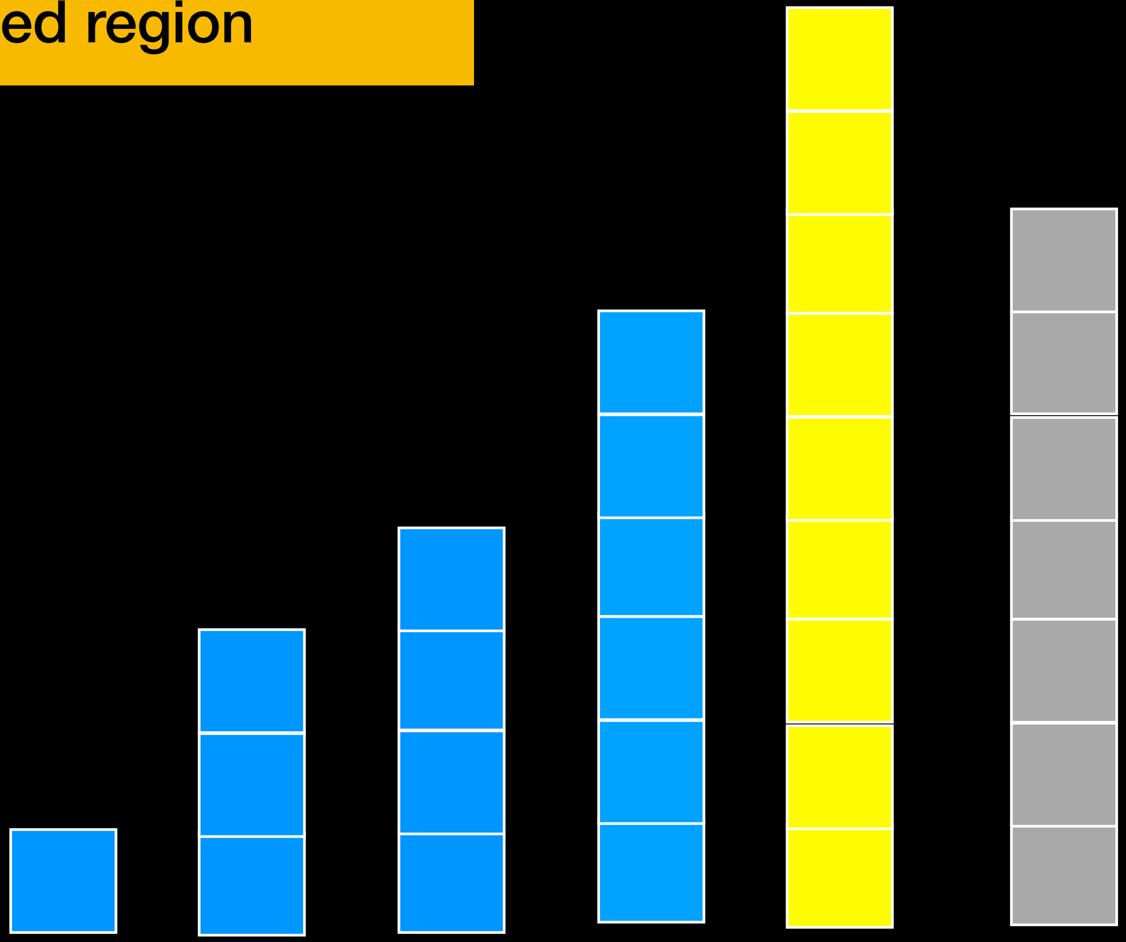
Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

4th Pass



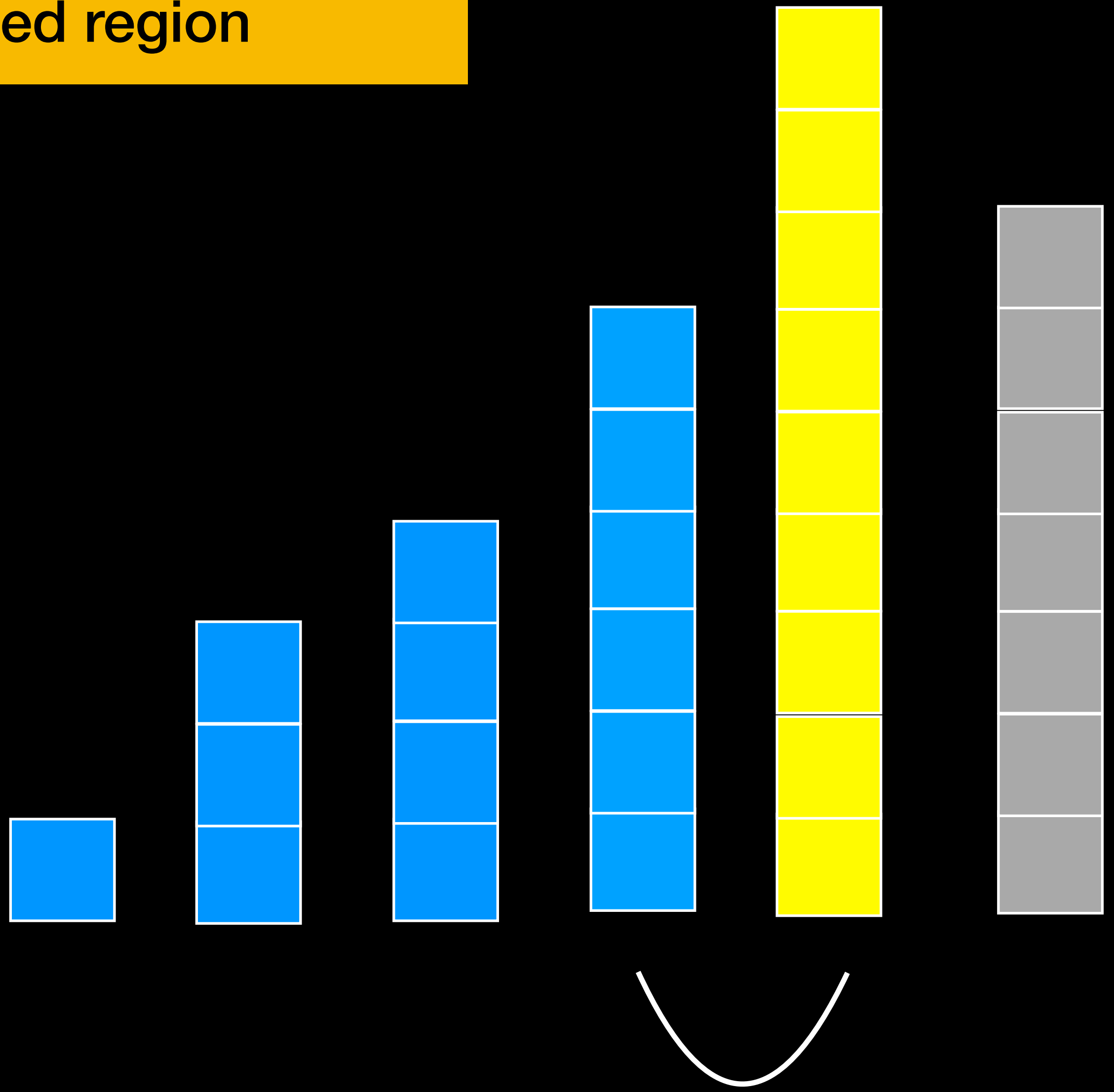
Insertion Sort

▨ Unsorted

🟦 Sorted

Pick first element in unsorted region and put it in right place in sorted region

4th Pass

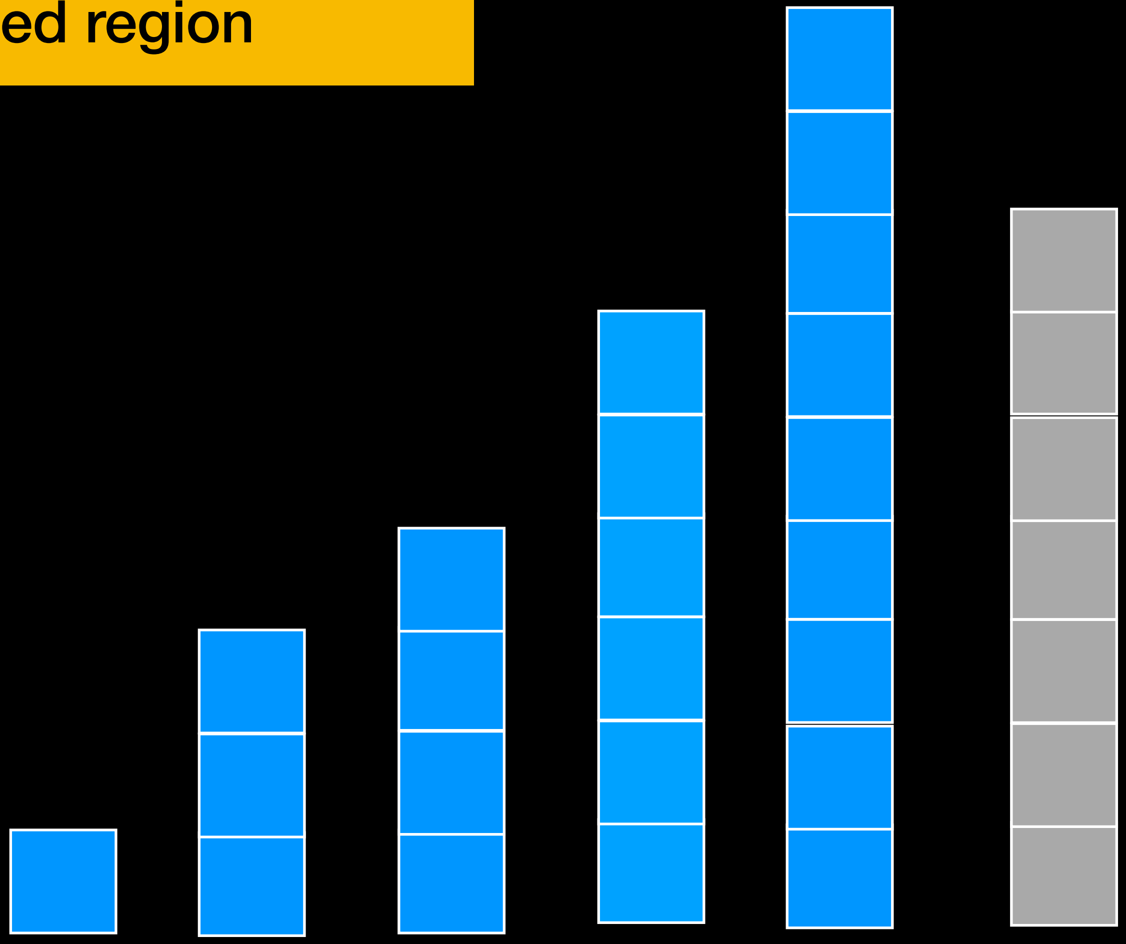


Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort

⬜ **Unsorted**

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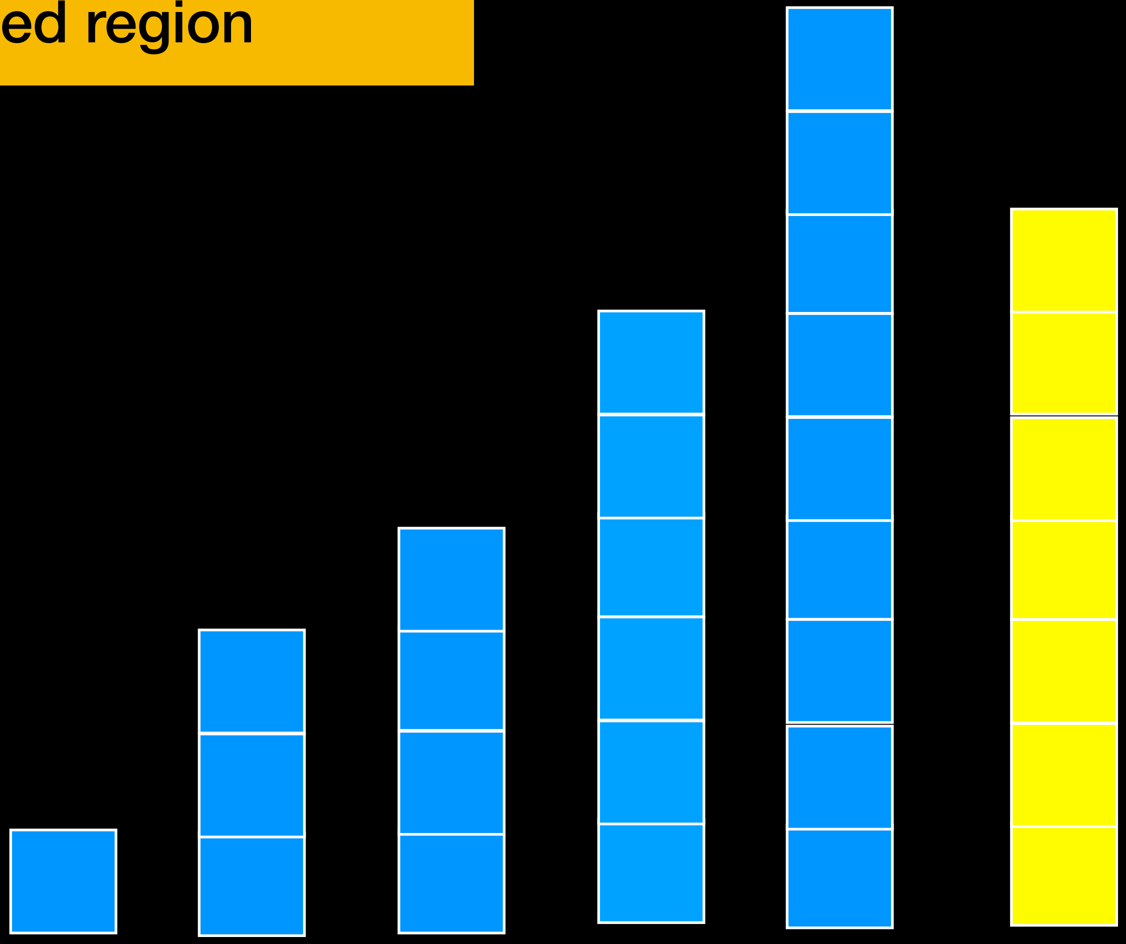
Pick first element in unsorted region and put it in right place in sorted region

5th Pass



Insertion Sort

▨ **Unsorted**

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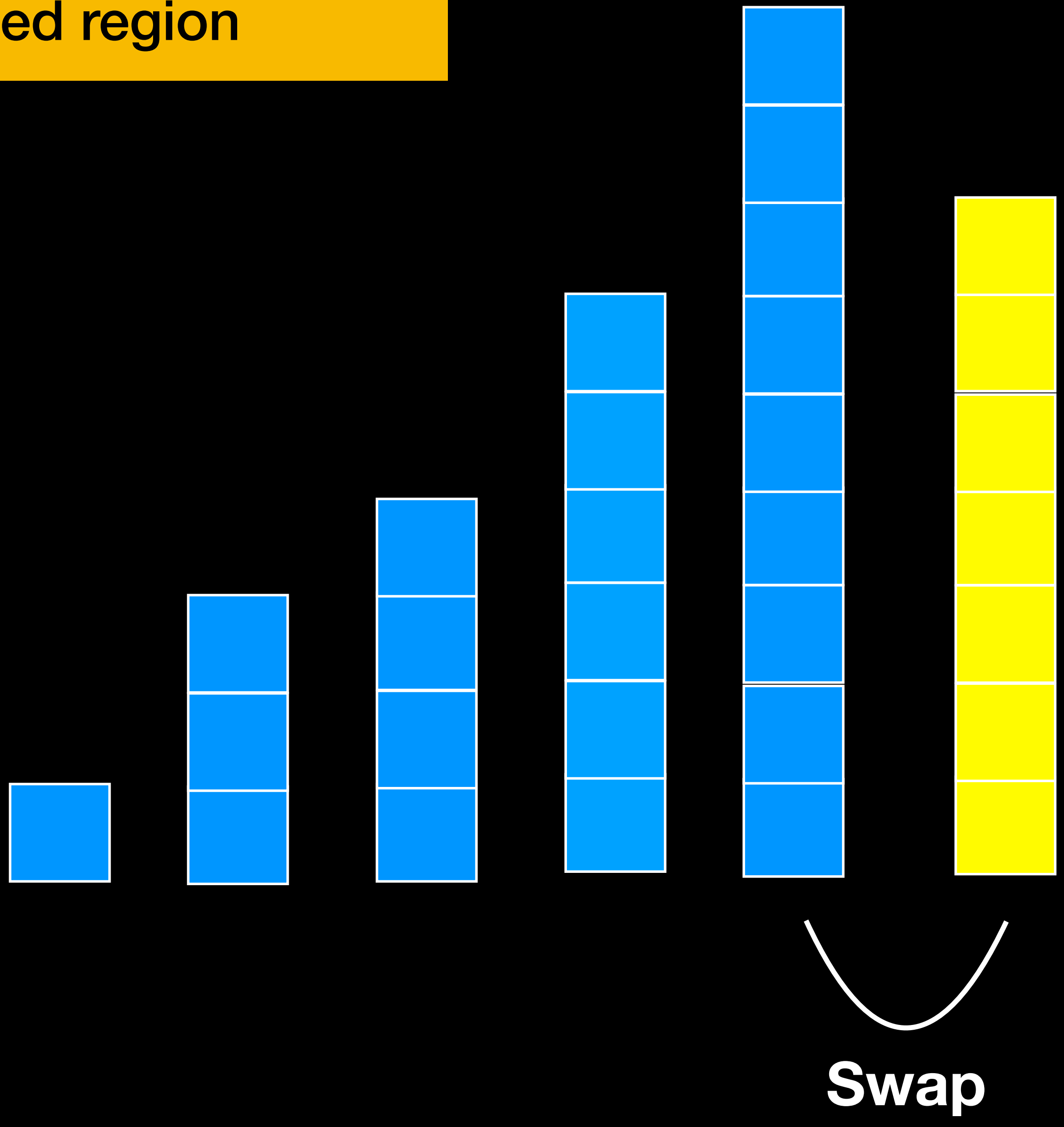
Pick first element in unsorted region and put it in right place in sorted region

5th Pass



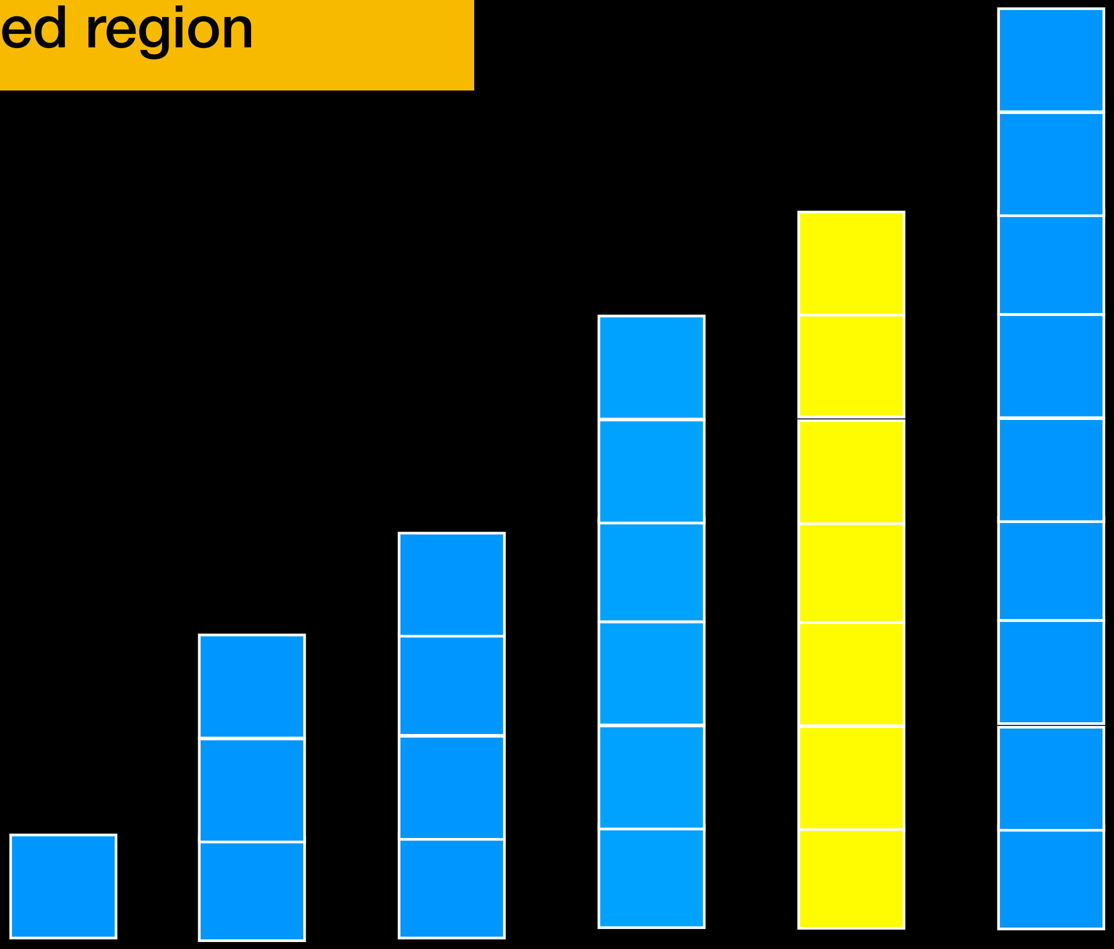
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

▨ **Unsorted**

🟦 **Sorted**

5th Pass



Insertion Sort

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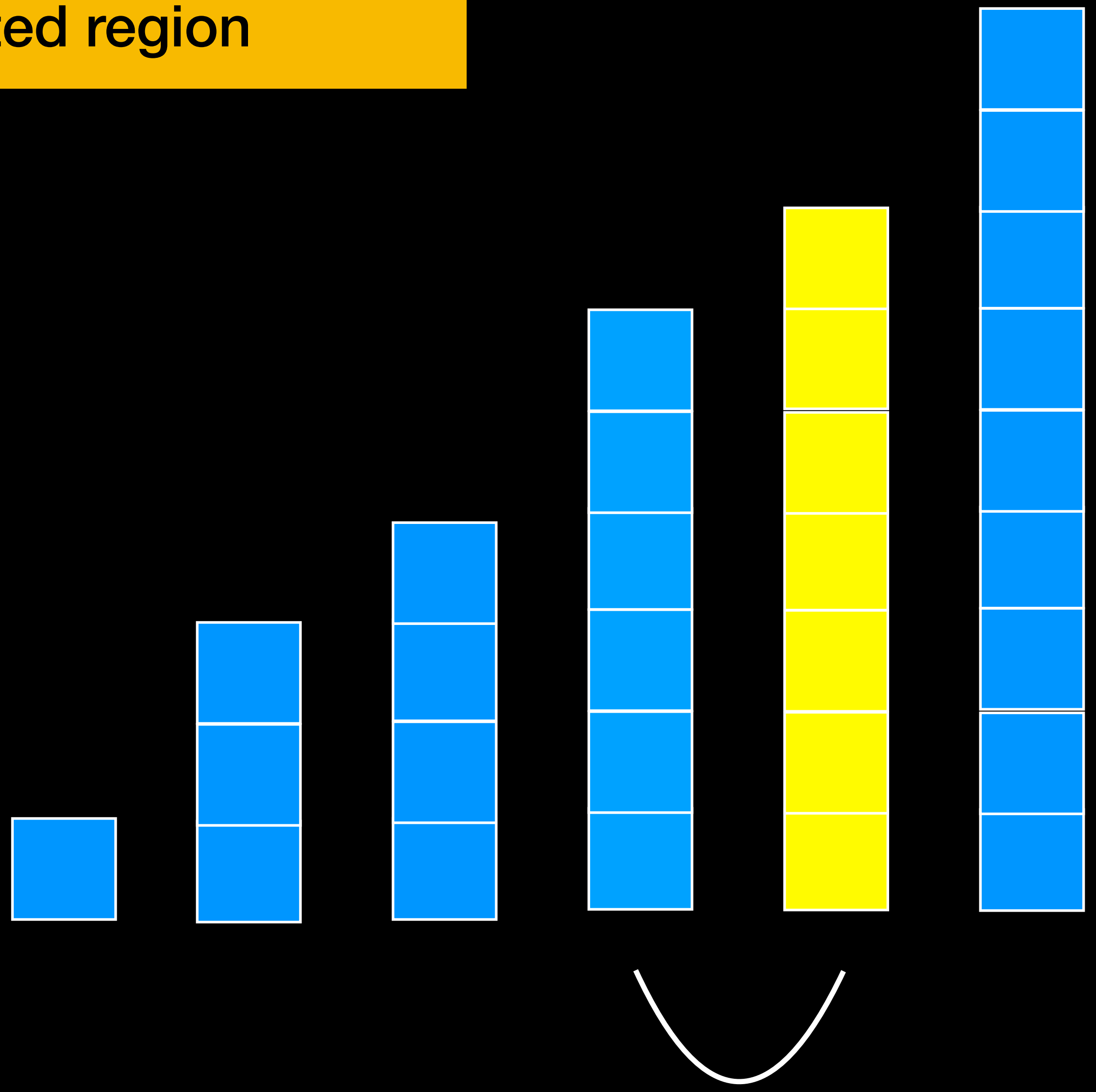
5th Pass



Insertion Sort

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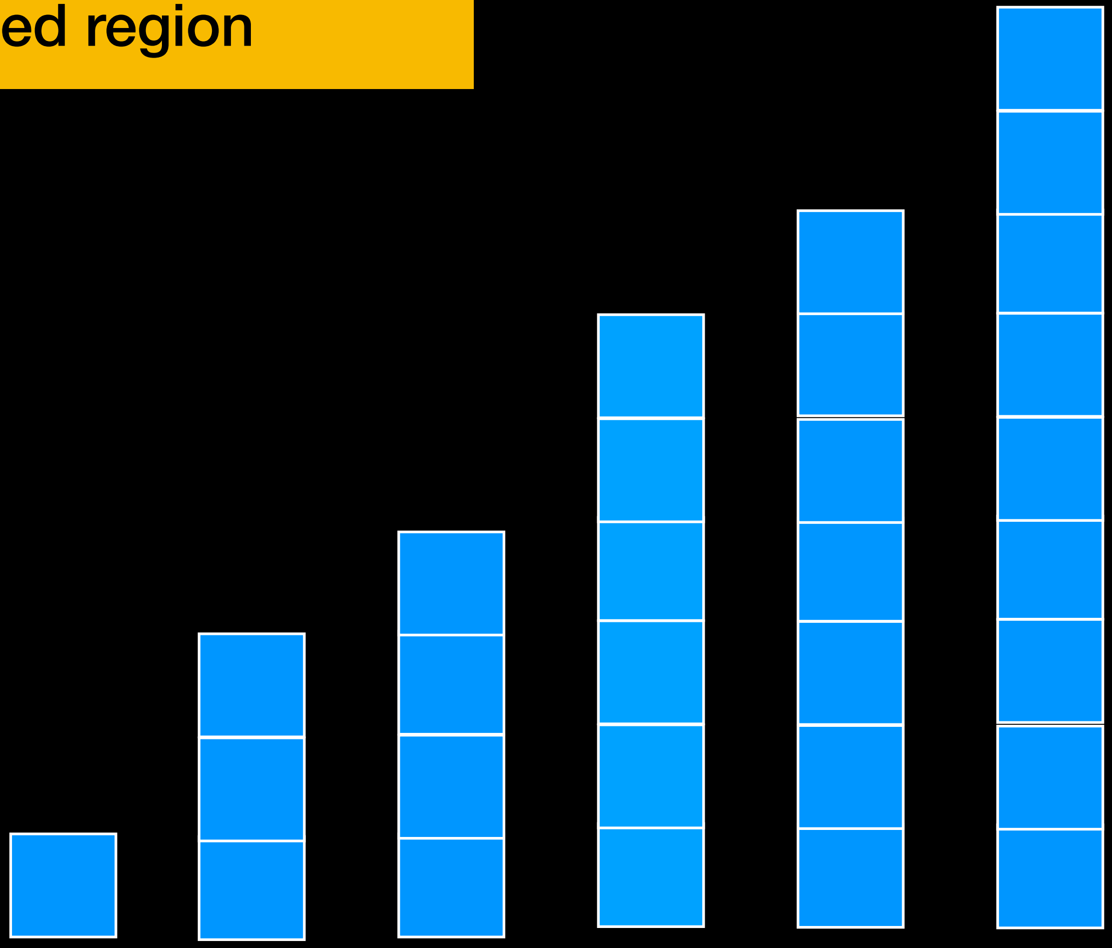


Insertion Sort Analysis

How much work?

First pass: **1** comparison and at most **1** swap

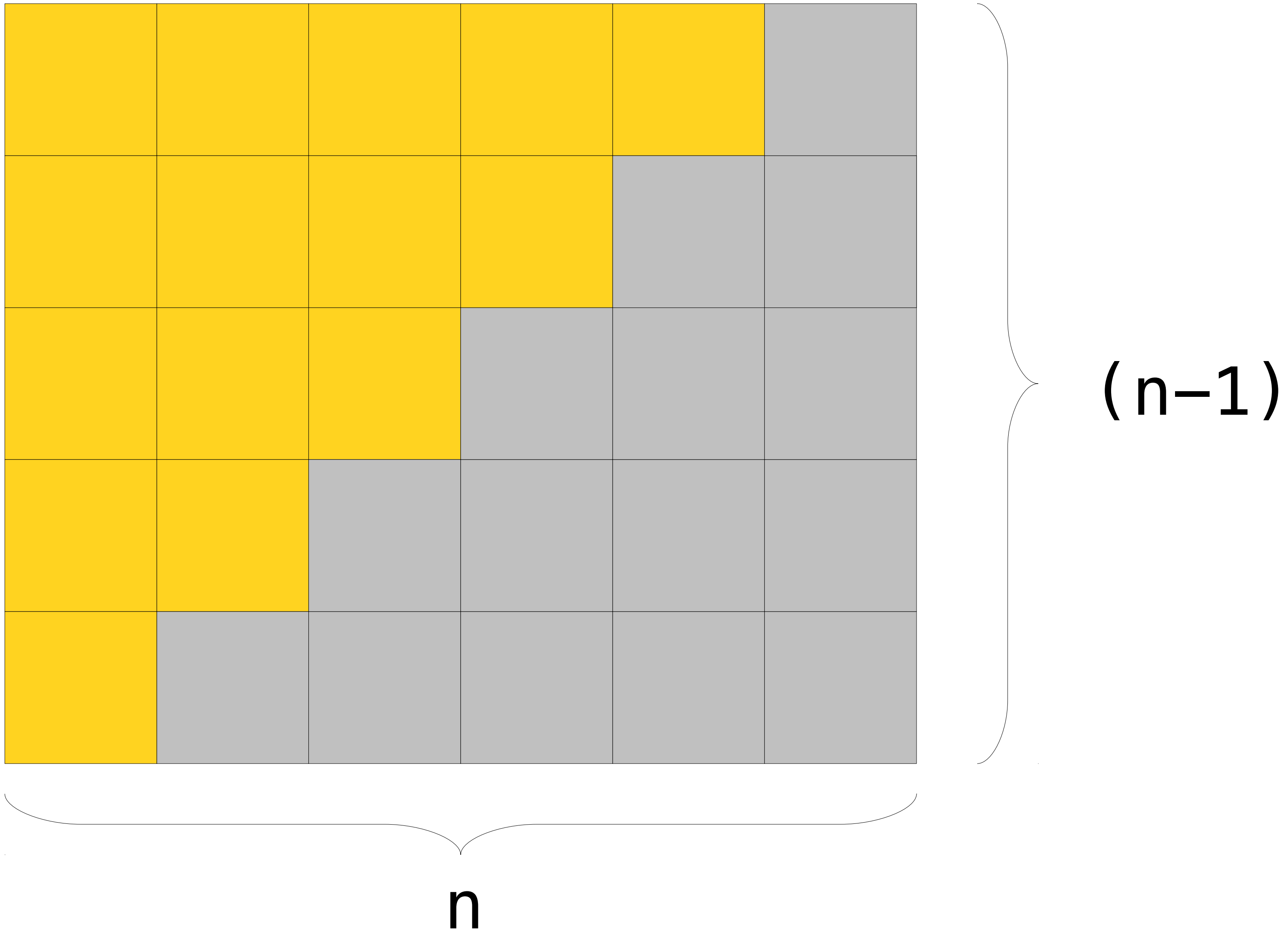
Second pass: at most **2** comparisons and at most **2** swaps

Third pass: at most **3** comparisons and at most **3** swaps

. . .

Total work: **1 + 2 + 3 + . . . + (n-1)**

$1 + 2 + . . . (n-2) + (n-1) = n(n-1)/2$



Insertion Sort Analysis

$T(n) = n(n-1) / 2$ comparisons $+ n(n-1) / 2$ swaps $= O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ )= O(\ )$?

$T(n) = n^2-n = O(\ \mathbf{n^2})$

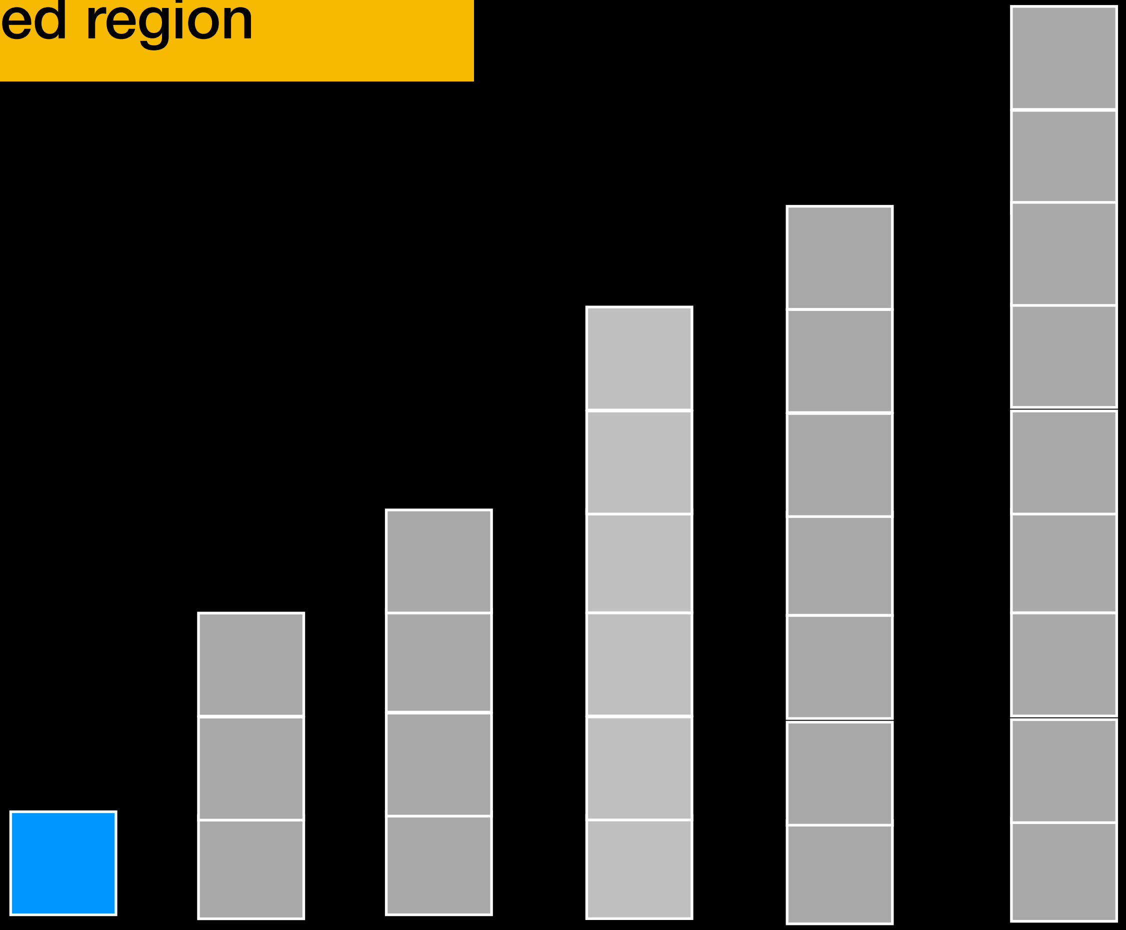
Insertion Sort run time is $\mathbf{O(n^2)}$

Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort

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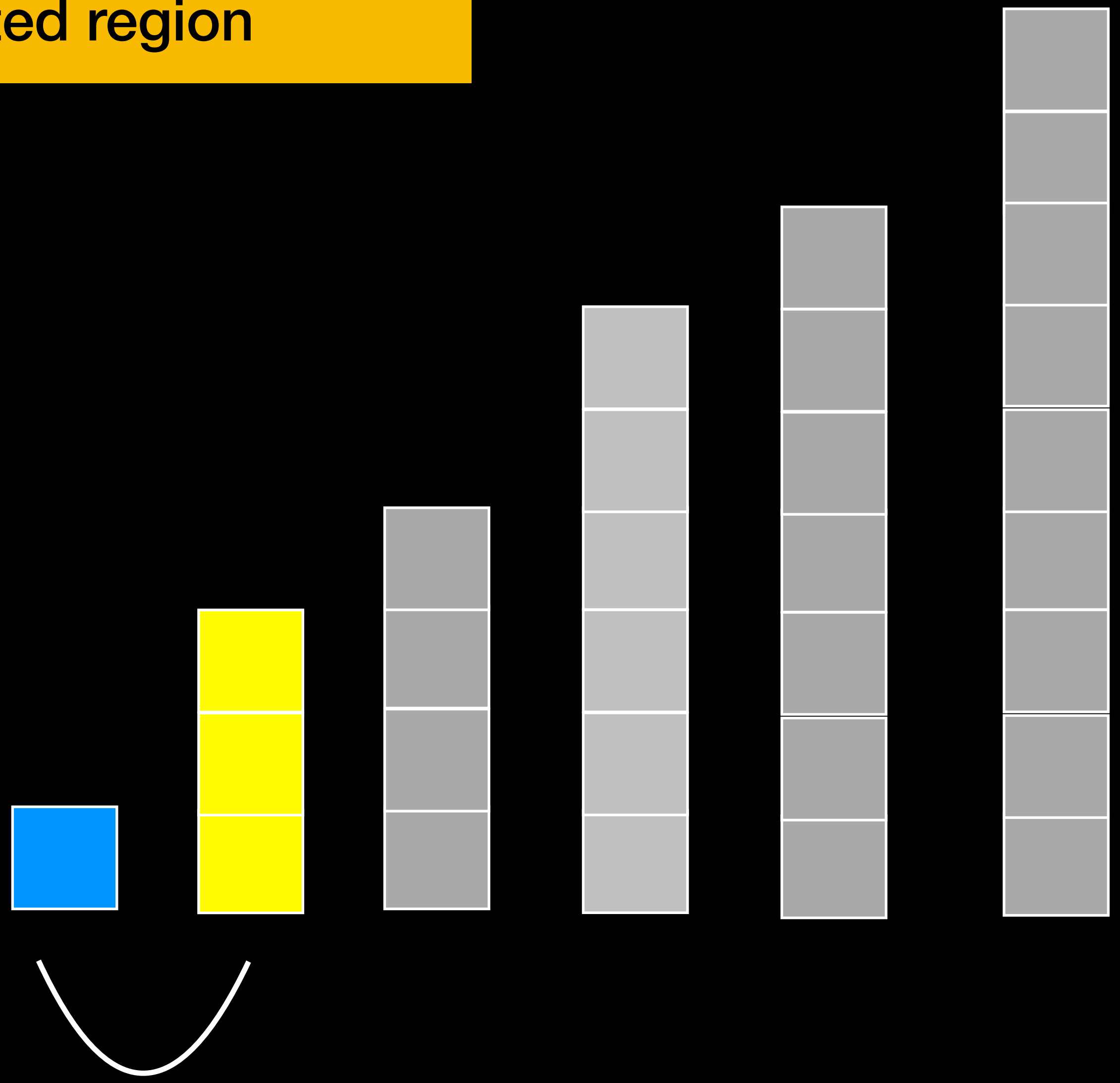

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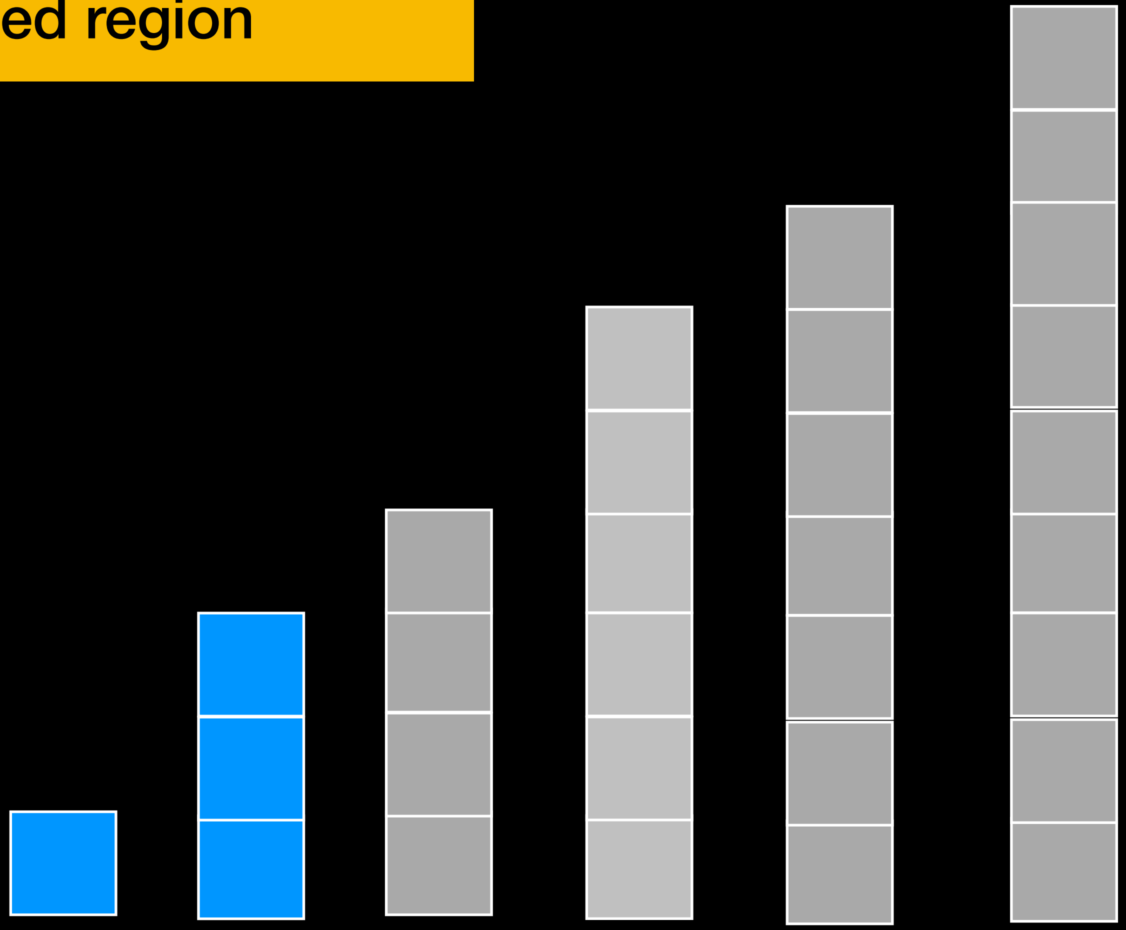

Insertion Sort

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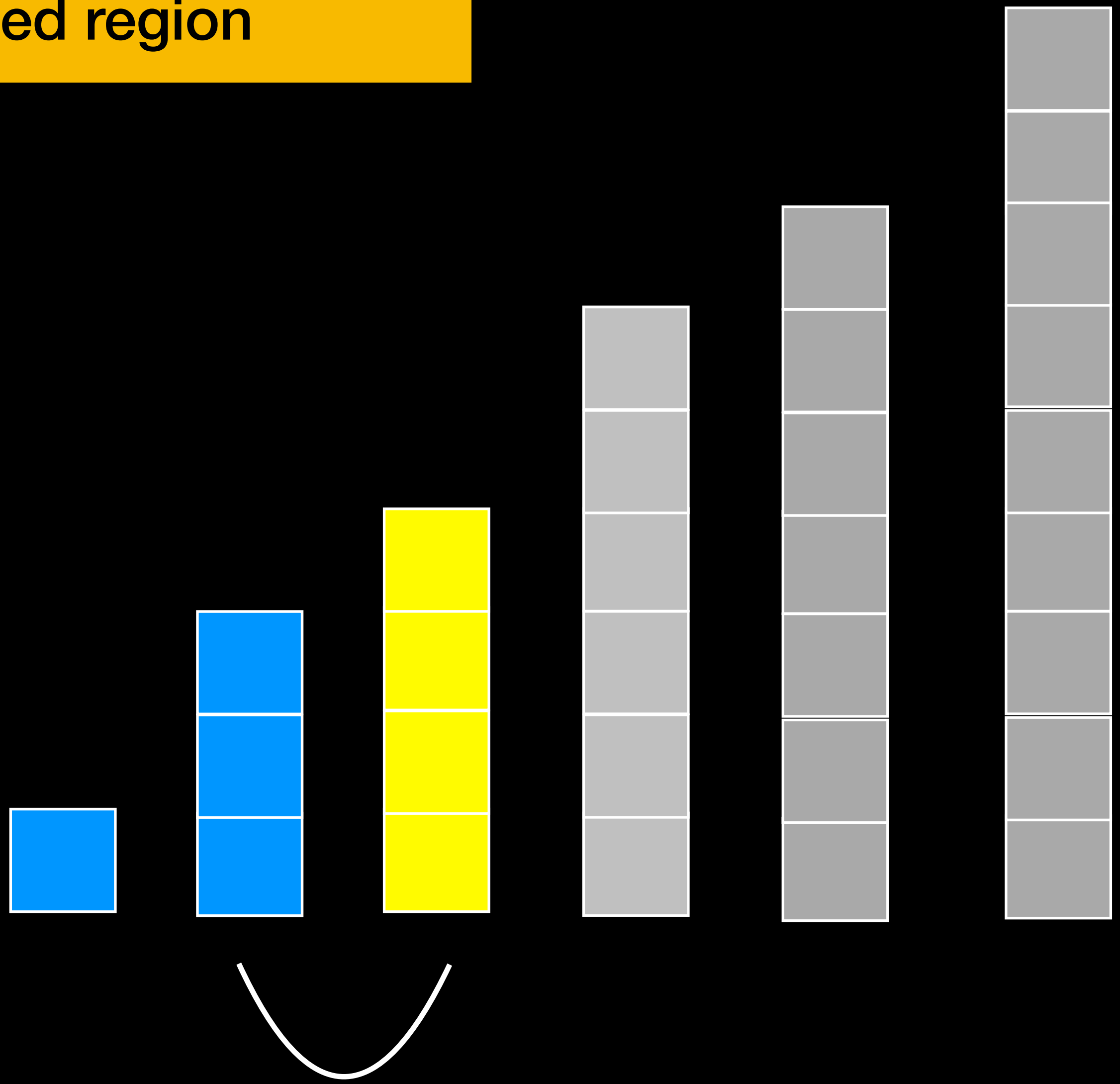
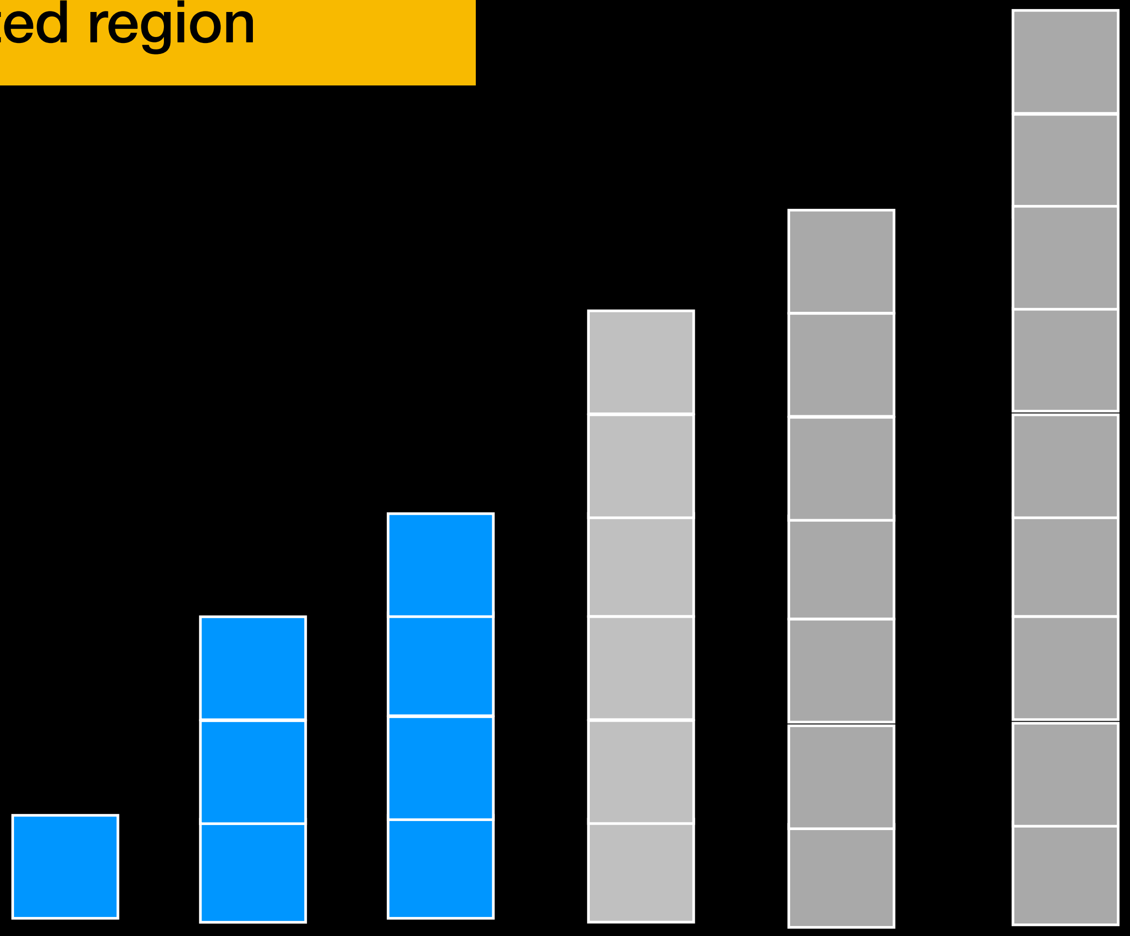


Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort

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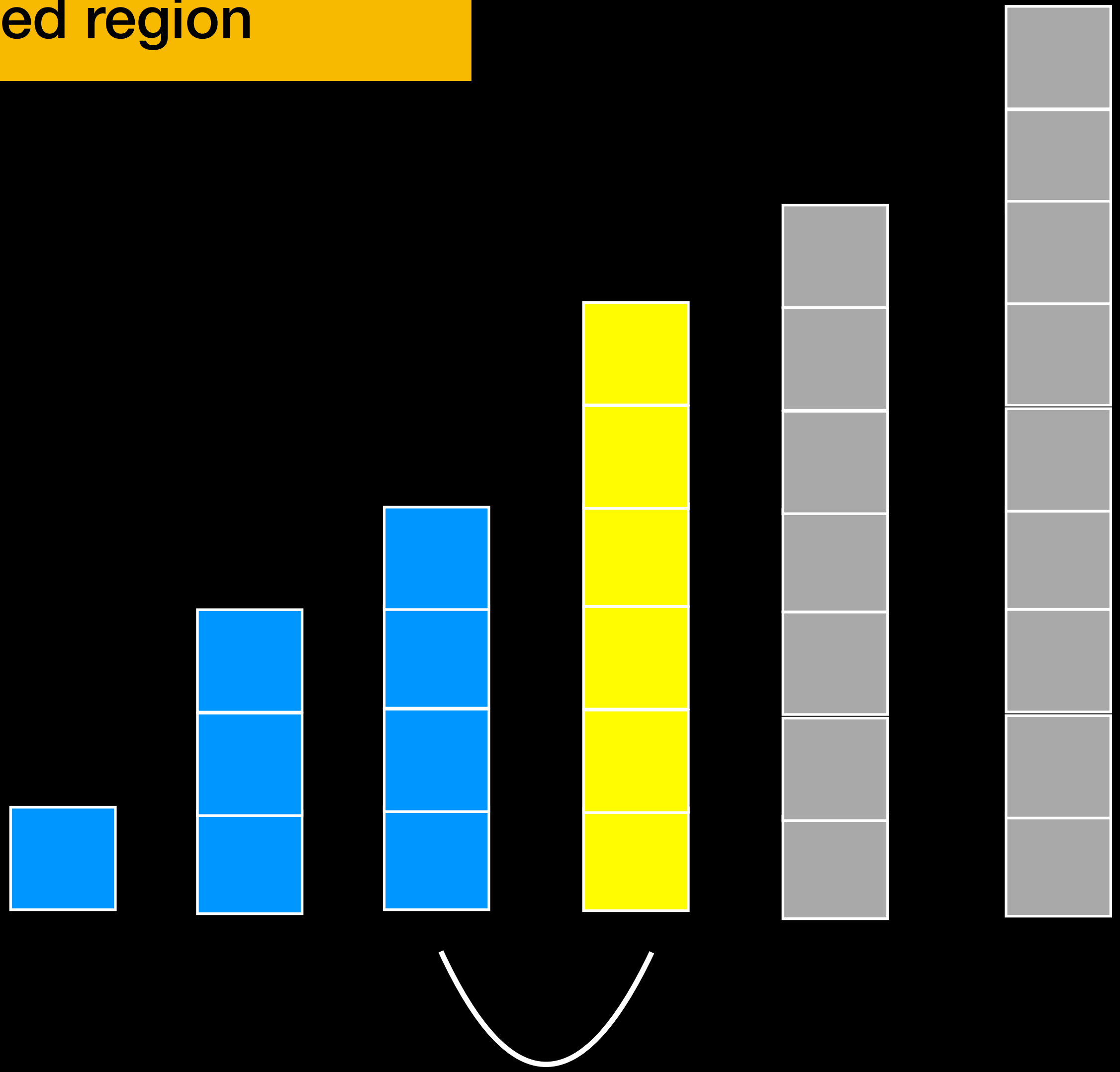

Insertion Sort

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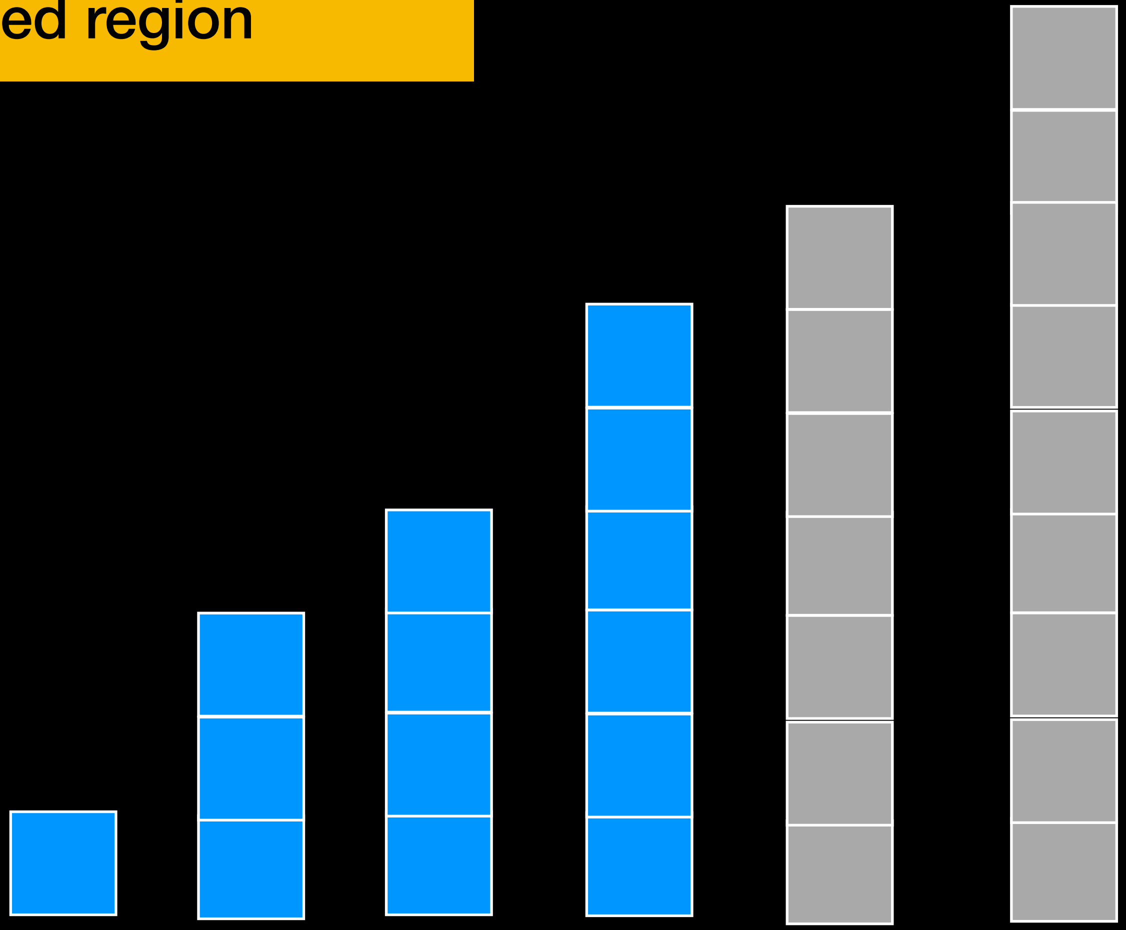

Insertion Sort

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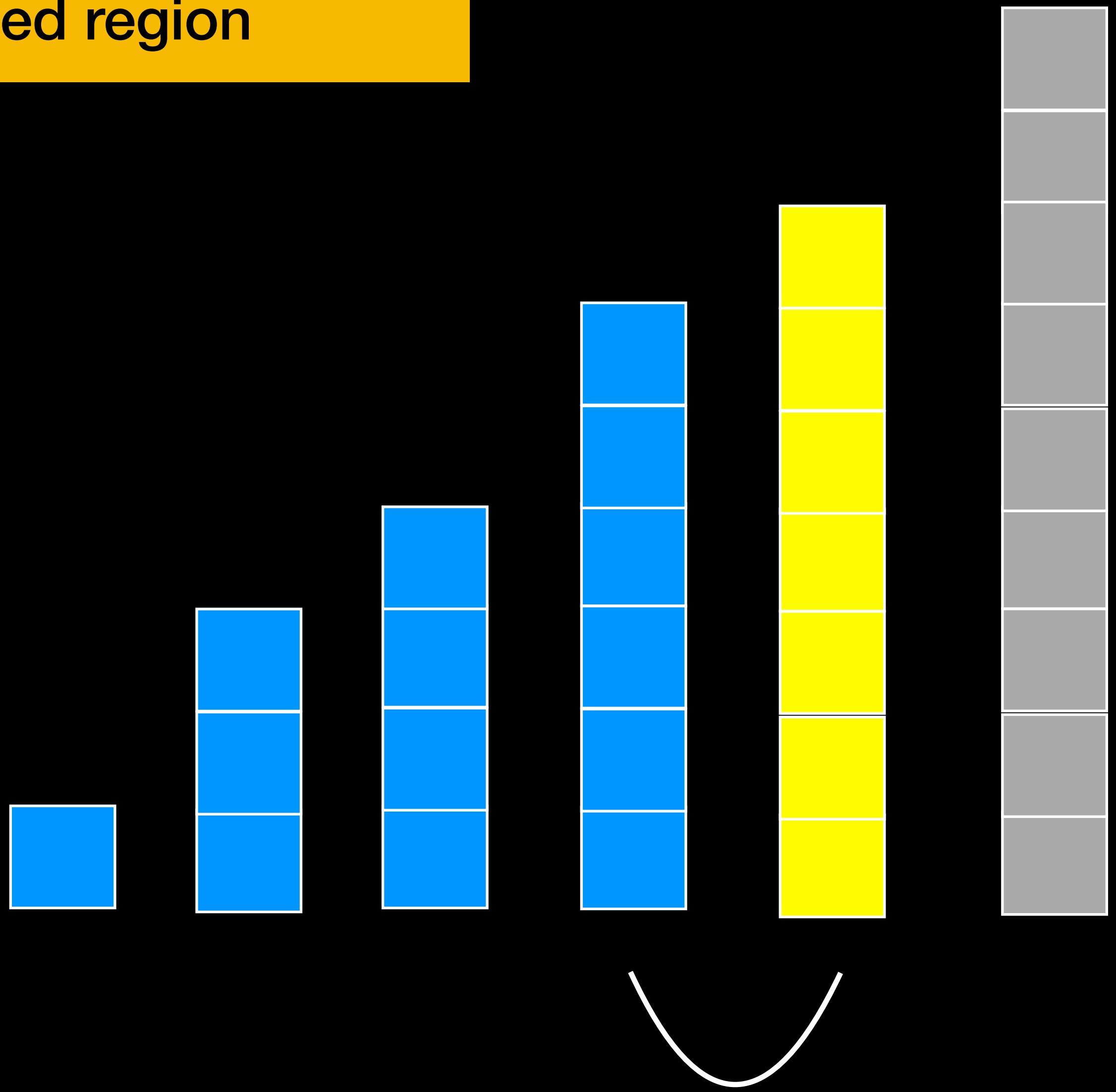
🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

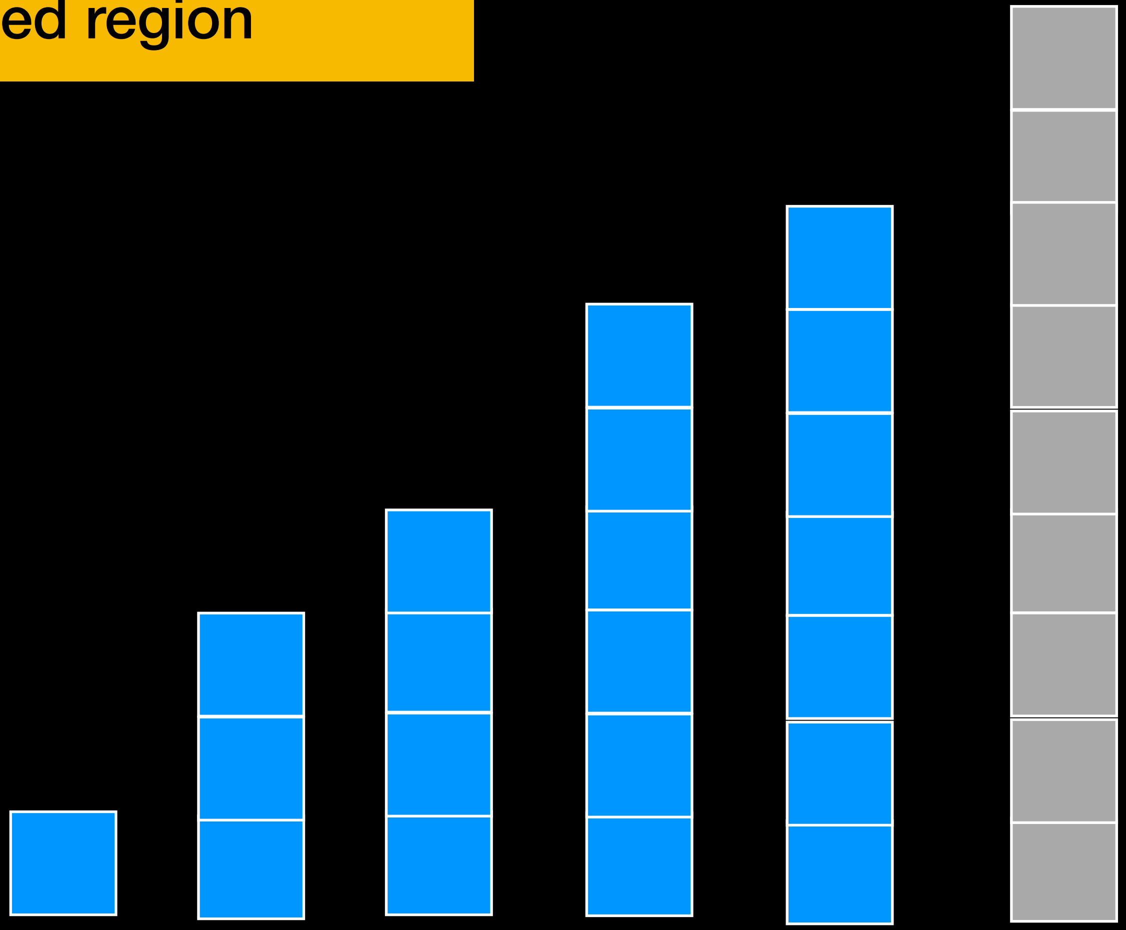


Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

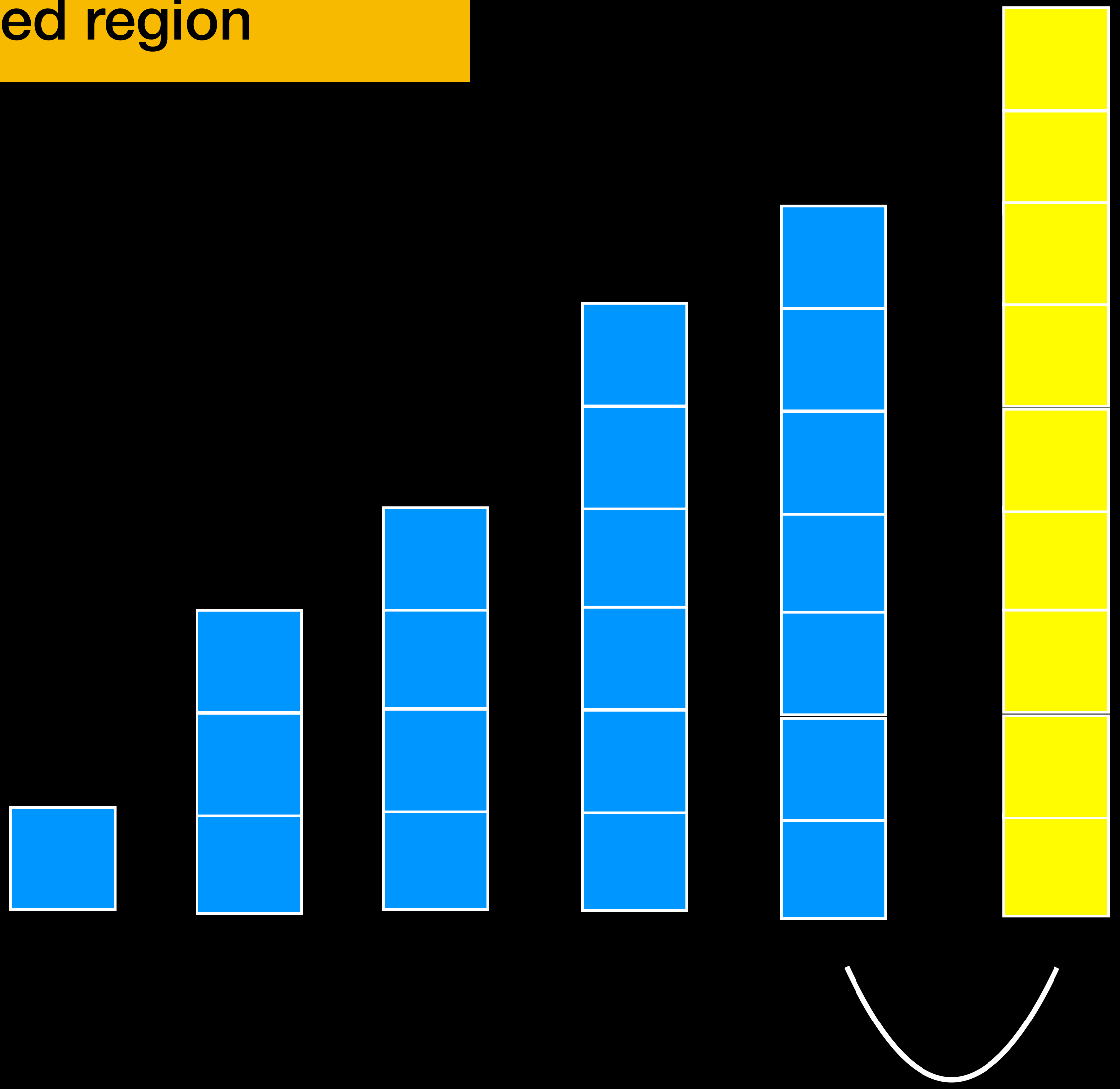


Insertion Sort

▨ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region

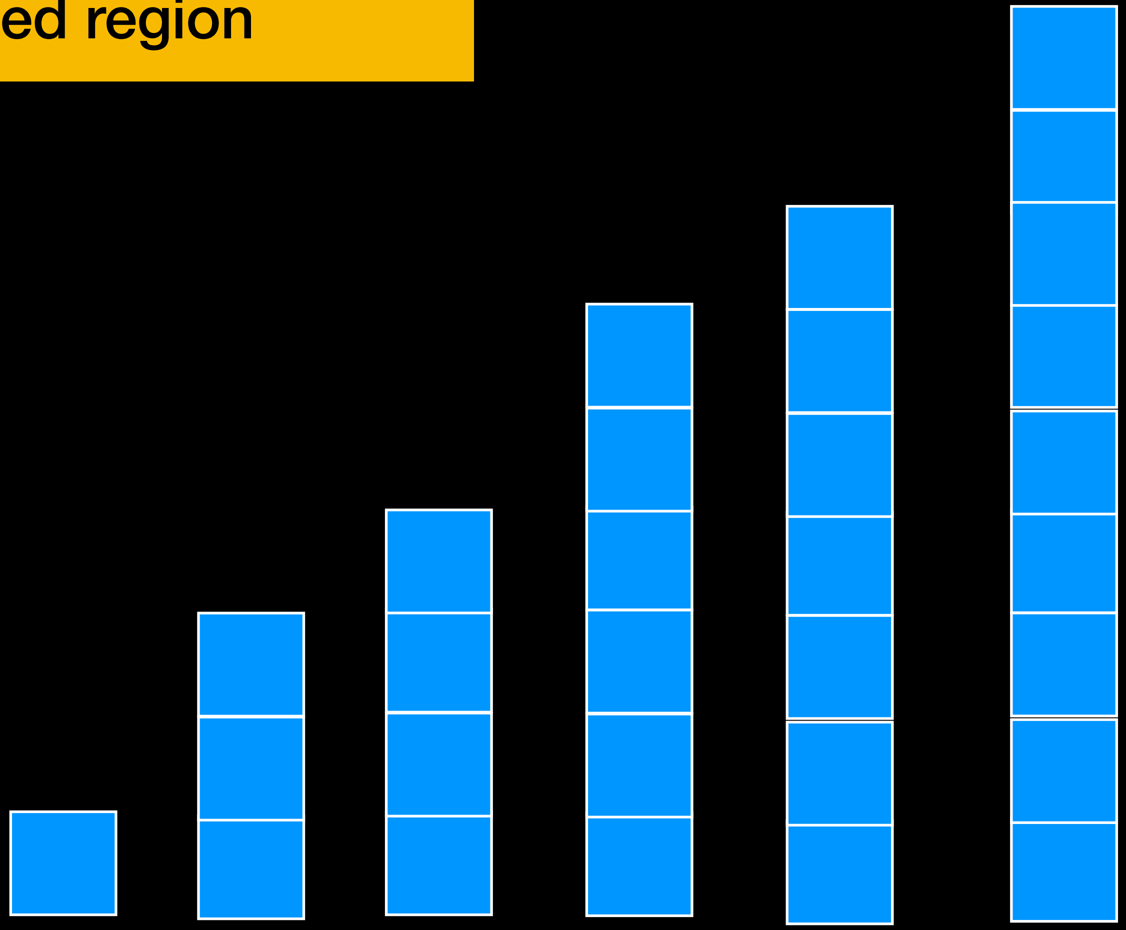


Insertion Sort

⬜ **Unsorted**

🟦 **Sorted**

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort Analysis

Execution time DOES depend on initial arrangement of data

Worst case: O(n^2) comparisons and data moves

Best case: O(n) comparisons and data moves

Stable

If array is already sorted Insertion sort will do only n comparisons and no swaps => good choice for small n and data likely somewhat sorted

```
template<typename ItemType>
void insertionSort(ItemType the_array[], size_t size)
{
   // unsorted = first index of the unsorted region,
   // Initially, sorted region is the_array[0],
   // unsorted region is the_array[1 ... size-1].
   // In general, sorted region is the_array[0 ... unsorted-1],
   // unsorted region the_array[unsorted ... size-1]
   for (int unsorted = 1; unsorted < size; unsorted++)
   {
      // At this point, the_array[0 ... unsorted-1] is sorted.
      // Keep swapping item to be inserted currently at
      // the_array[unsorted] with items at lower indices
      // as long as its value is > the index of the item
      int current = unsorted; // currently being inserted
      while ((current > 0) &&
             (the_array[current - 1] > the_array[current]))
      {
         swap(the_array[current], the_array[current - 1]); // swap
         current--;
      }  // end while
   }  // end for
}  // end insertionSort
```

```
template<typename ItemType>
void insertionSort(ItemType the_array[], size_t size)
{
   // unsorted = first index of the unsorted region,
   // Initially, sorted region is the_array[0],
   // unsorted region is the_array[1 ... size-1].
   // In general, sorted region is the_array[0 ... unsorted-1],
   // unsorted region the_array[unsorted ... size-1]
   for (int unsorted = 1; unsorted < size; unsorted++)
   {
      // At this point, the_array[0 ... unsorted-1] is sorted.
      // Keep swapping item to be inserted currently at
      // the_array[unsorted] with items at lower indices
      // as long as its value is > the index of the item
      int current = unsorted; // currently being inserted
      while ((current > 0) &&
             (the_array[current - 1] > the_array[current]))
      {
         swap(the_array[current], the_array[current - 1]); // swap
         current--;
      }  // end while
   }  // end for
}  // end insertionSort
```

Pass O(n)

O(n)

$O(n^2)$

Raise your hand if you had Insertion Sort

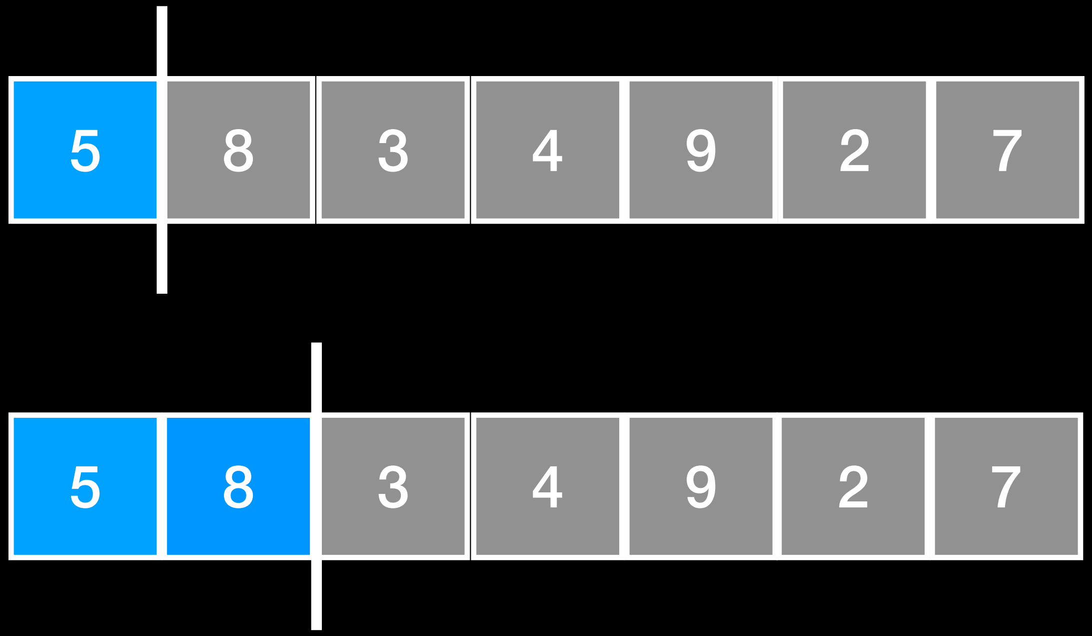
What we have so far

	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Bubble Sort	$O(n^2)$	$O(n)$
Insertion Sort	$O(n^2)$	$O(n)$

Pick first element in unsorted region and put it in right place in sorted region

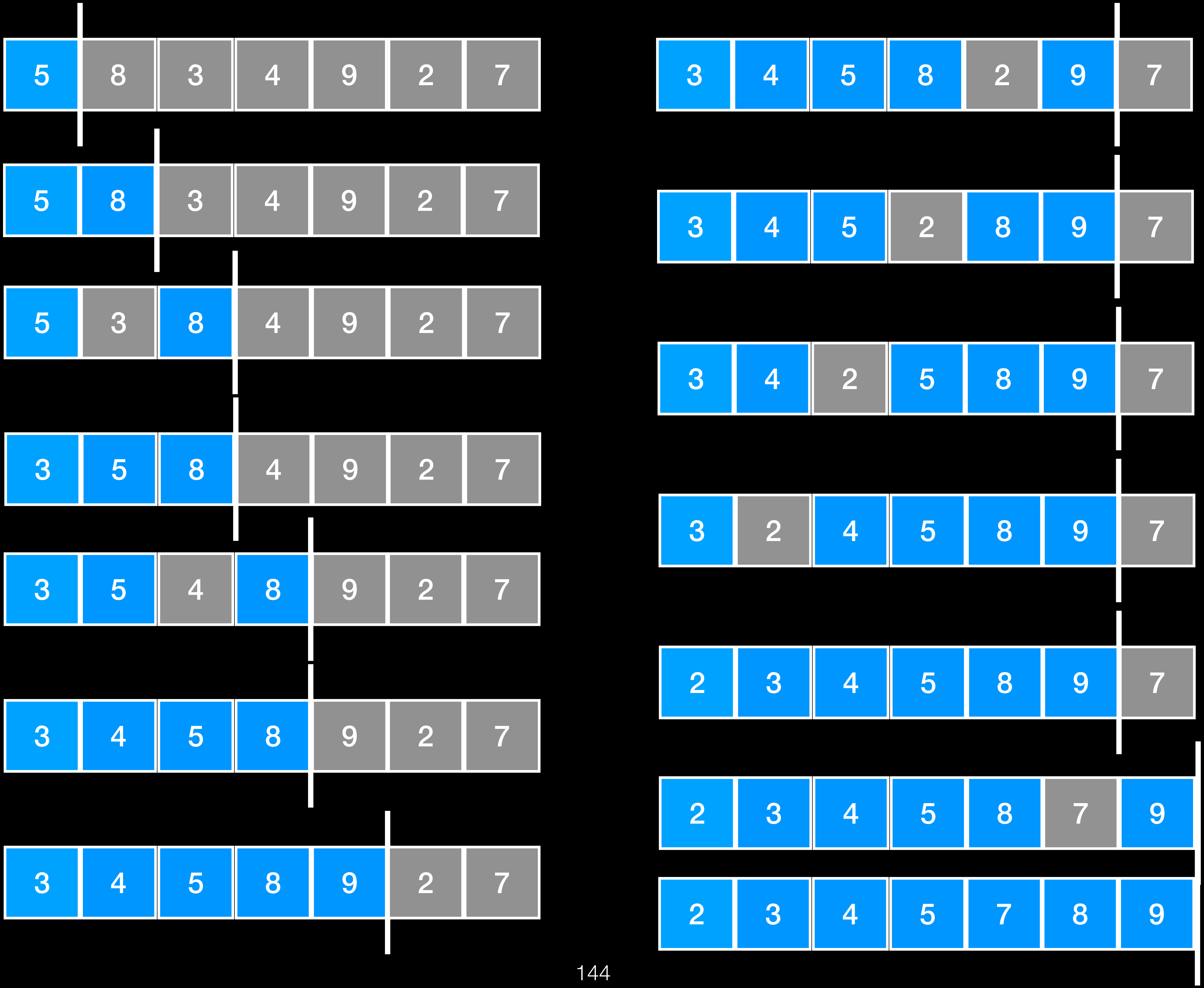
Lecture Activity

Sort the array using **Insertion Sort**
Show the entire array after each comparison/swap operation and at each step mark clearly the division between the sorted and unsorted portions of the array



5	8	3	4	9	2	7

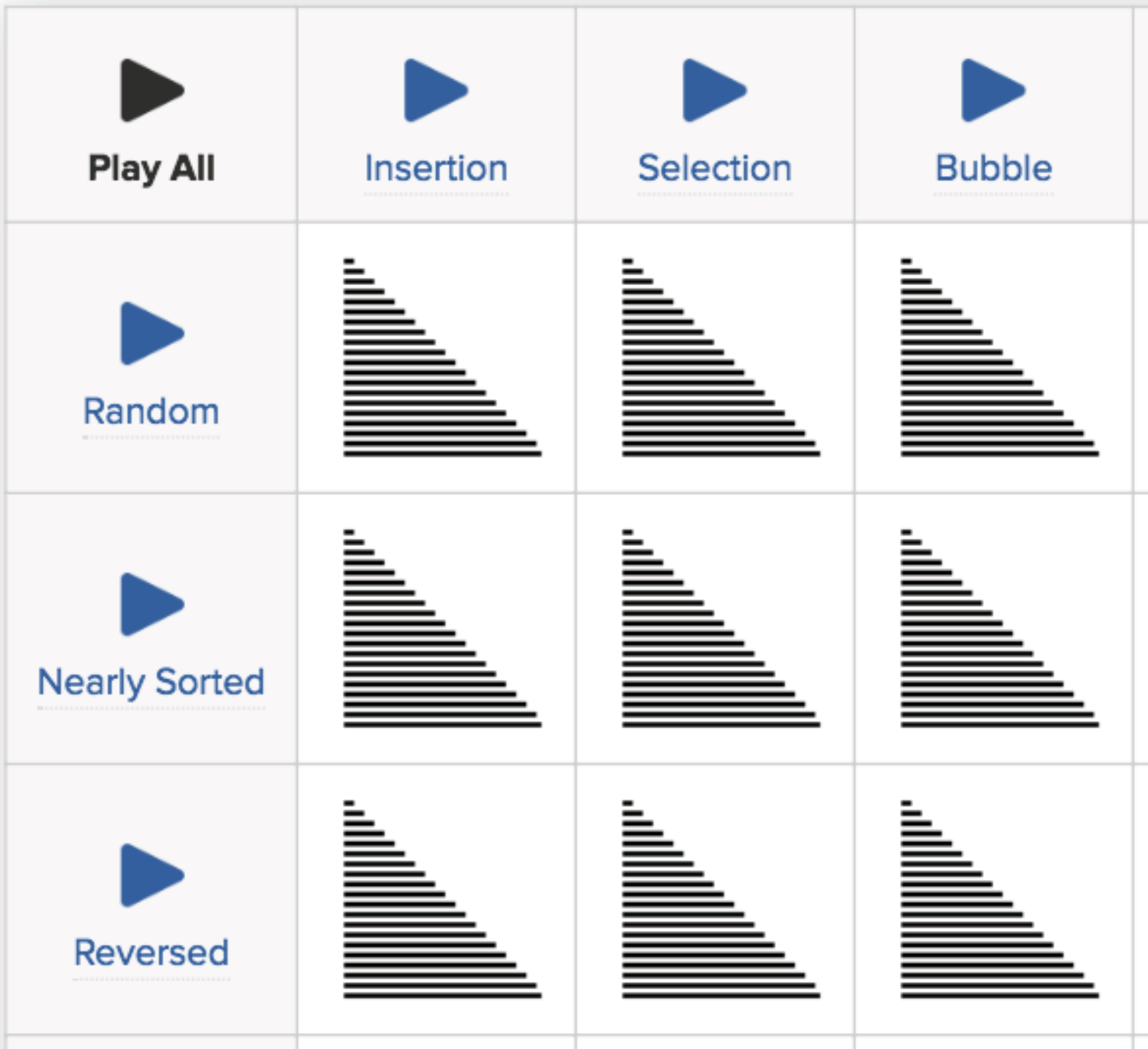
5	8	3	4	9	2	7



5	8	3	4	9	2	7
5	8	3	4	9	2	7
5	3	8	4	9	2	7
3	5	8	4	9	2	7
3	5	4	8	9	2	7
3	4	5	8	9	2	7
3	4	5	8	9	2	7

3	4	5	8	2	9	7
3	4	5	2	8	9	7
3	4	2	5	8	9	7
3	2	4	5	8	9	7
2	3	4	5	8	9	7
2	3	4	5	8	7	9
2	3	4	5	7	8	9

https://www.toptal.com/developers/sorting-algorithms



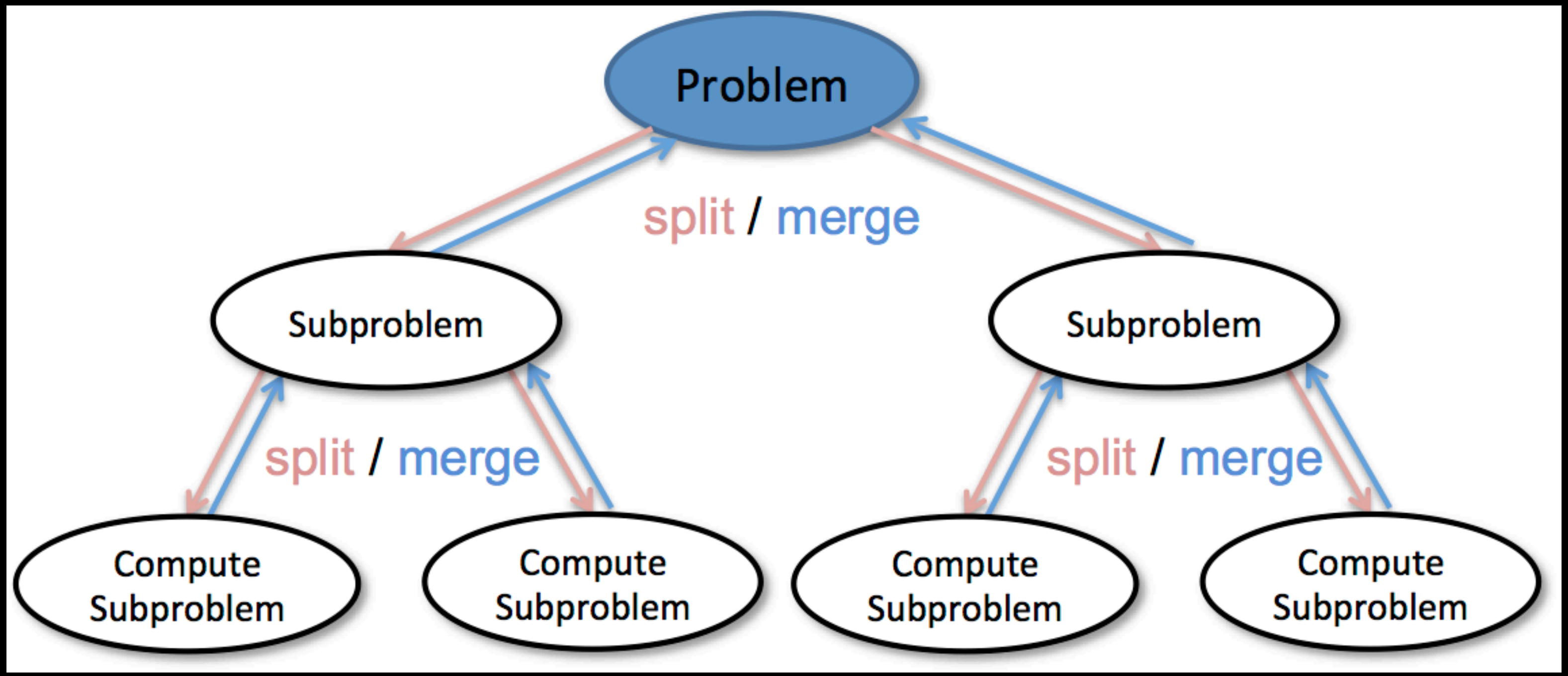
What we have so far

	Worst Case	**Best Case**
Selection Sort	$O(n^2)$	$O(n^2)$
Bubble Sort	$O(n^2)$	$O(n)$
Insertion Sort	$O(n^2)$	$O(n)$

Can we do better?

Can we do better?

Divide and Conquer!!!



Merge Sort

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

Understanding O(n^2)

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

$T(^1/_2 n)$

195	599	158	2	260	11	64	932	5

$T(^1/_2 n)$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

$T({}^1/_2 n)$

$T({}^1/_2 n)$

$$(n/2)^2 = n^2/4$$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

$T(^1/_2 n) \approx {}^1/_4\, T(n)$

$T(^1/_2 n) \approx {}^1/_4\, T(n)$

$(n/2)^2 = n^2/4$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

3	14	43	76	100	108	200	274	523

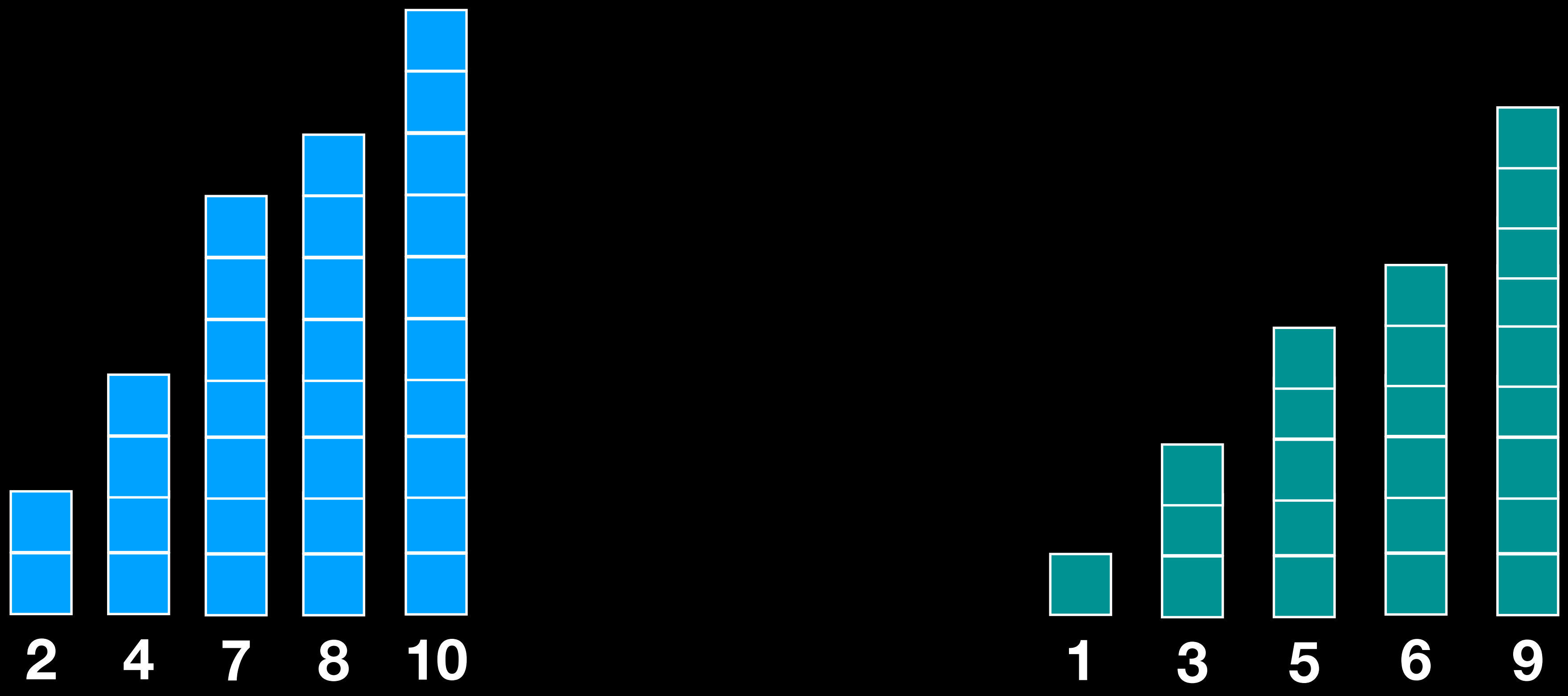
2	5	11	64	158	195	260	599	932

$T(\frac{1}{2}n) \approx \frac{1}{4}\,T(n)$

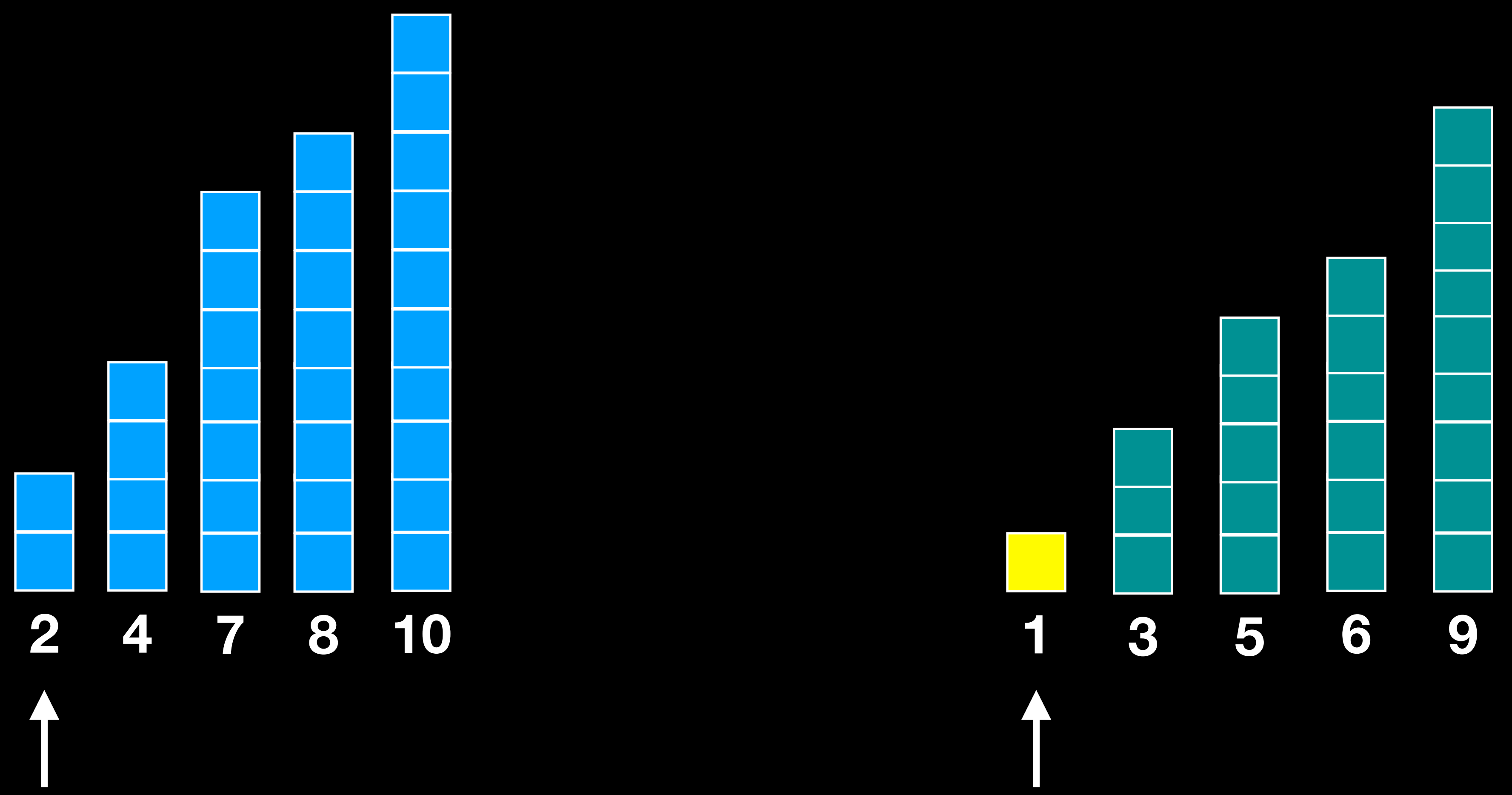
$T(\frac{1}{2}n) \approx \frac{1}{4}\,T(n)$

$$(n/2)^2 = n^2/4$$

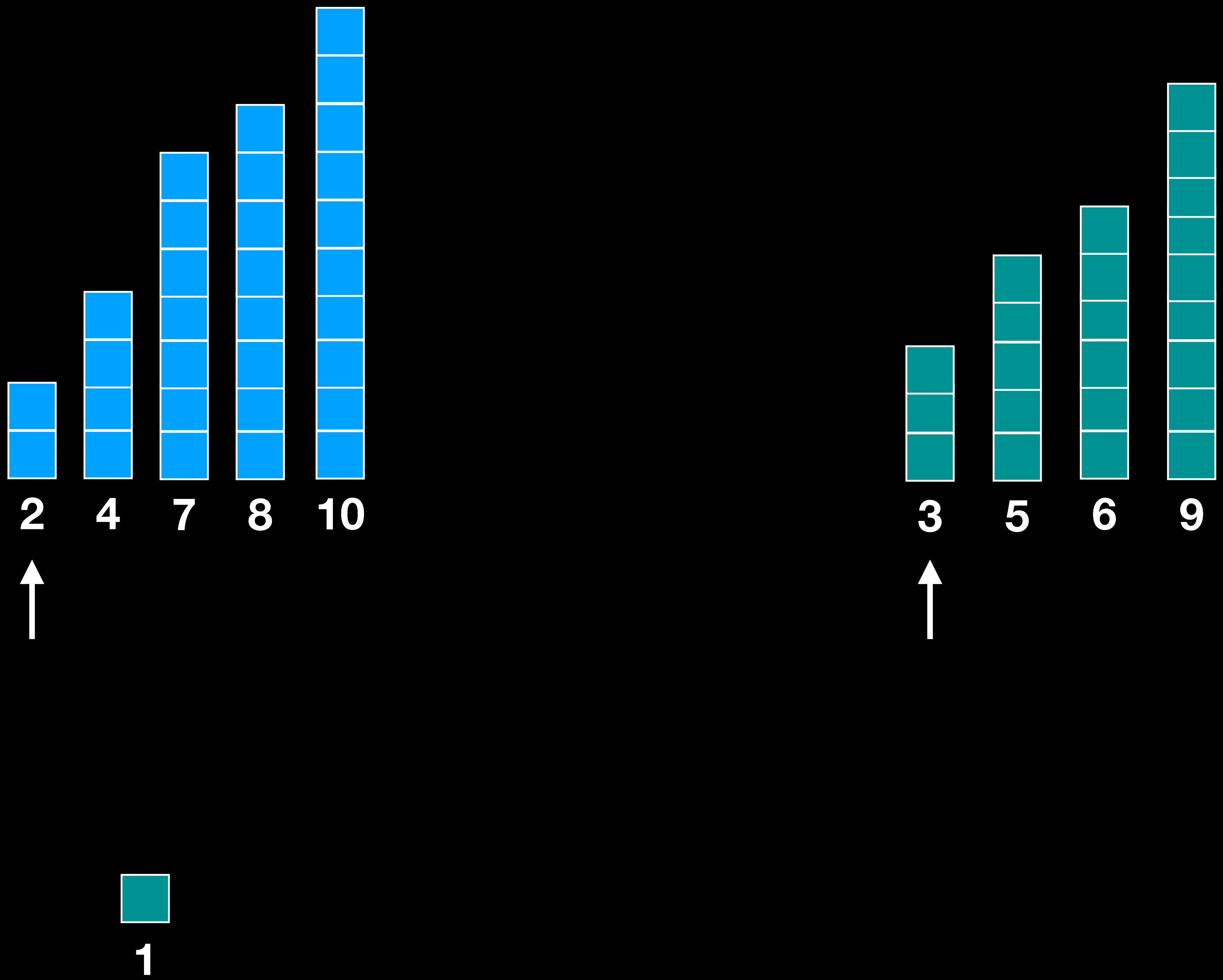
Key Insight: Merge is linear



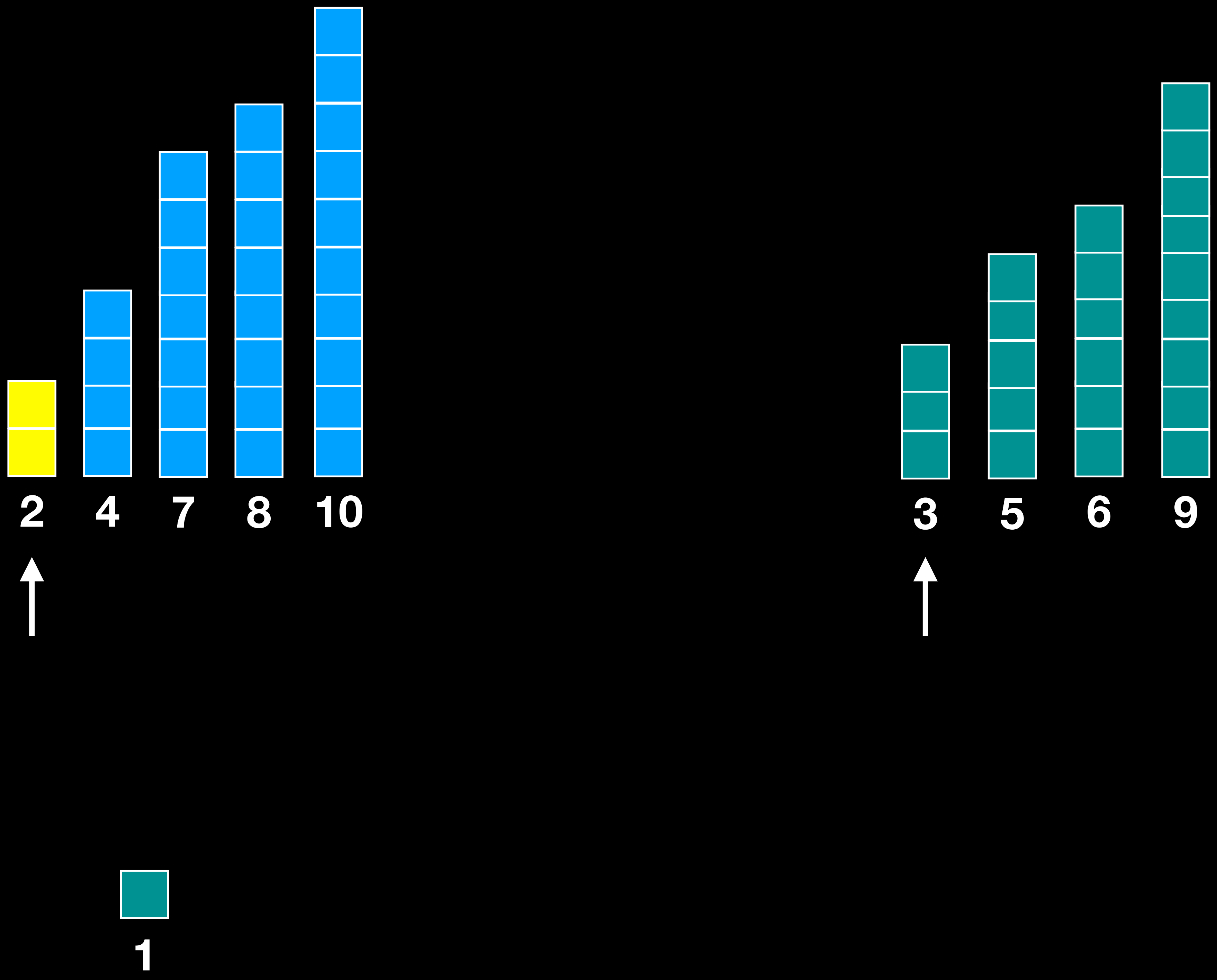
Key Insight: Merge is linear



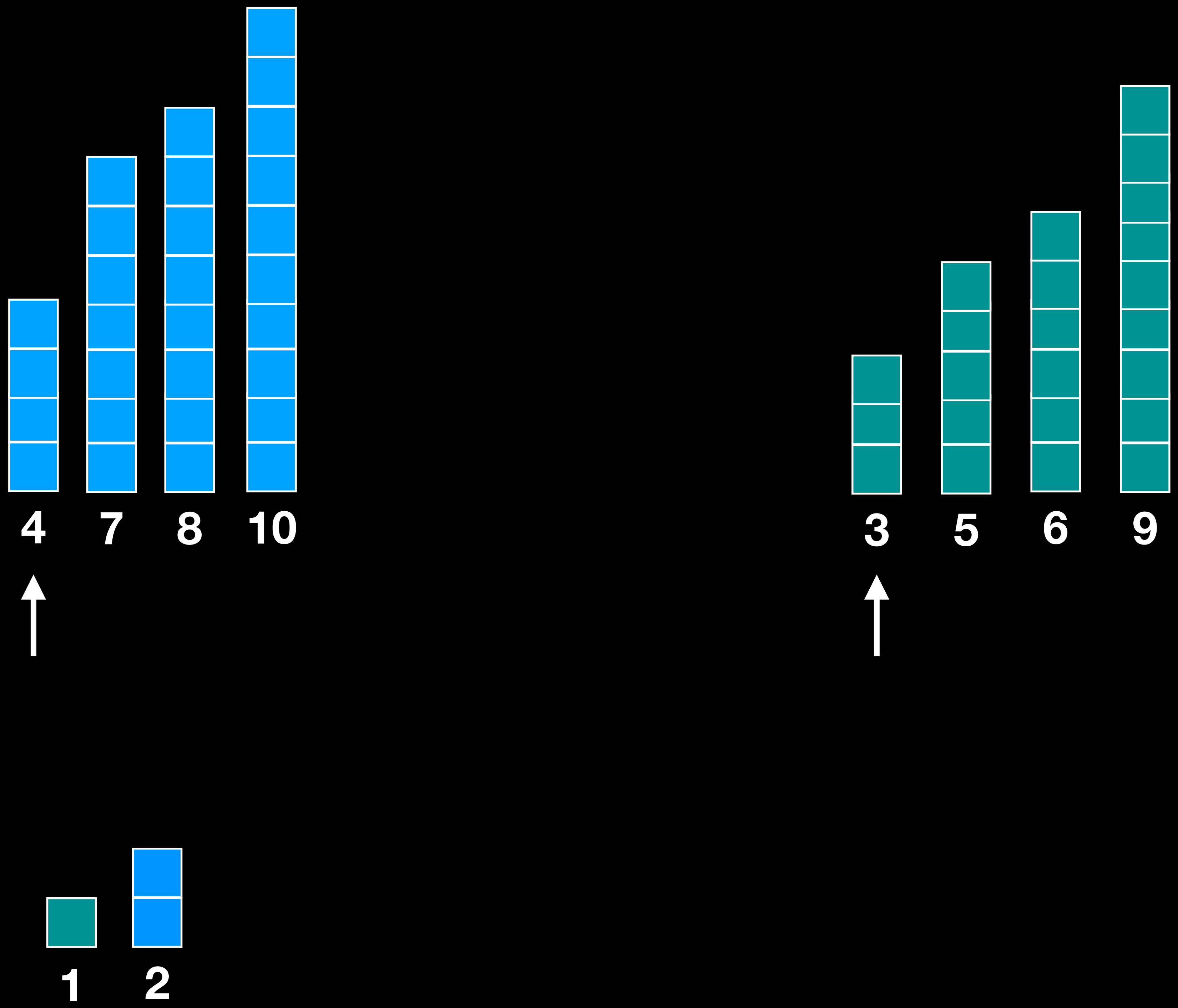
Key Insight: Merge is linear



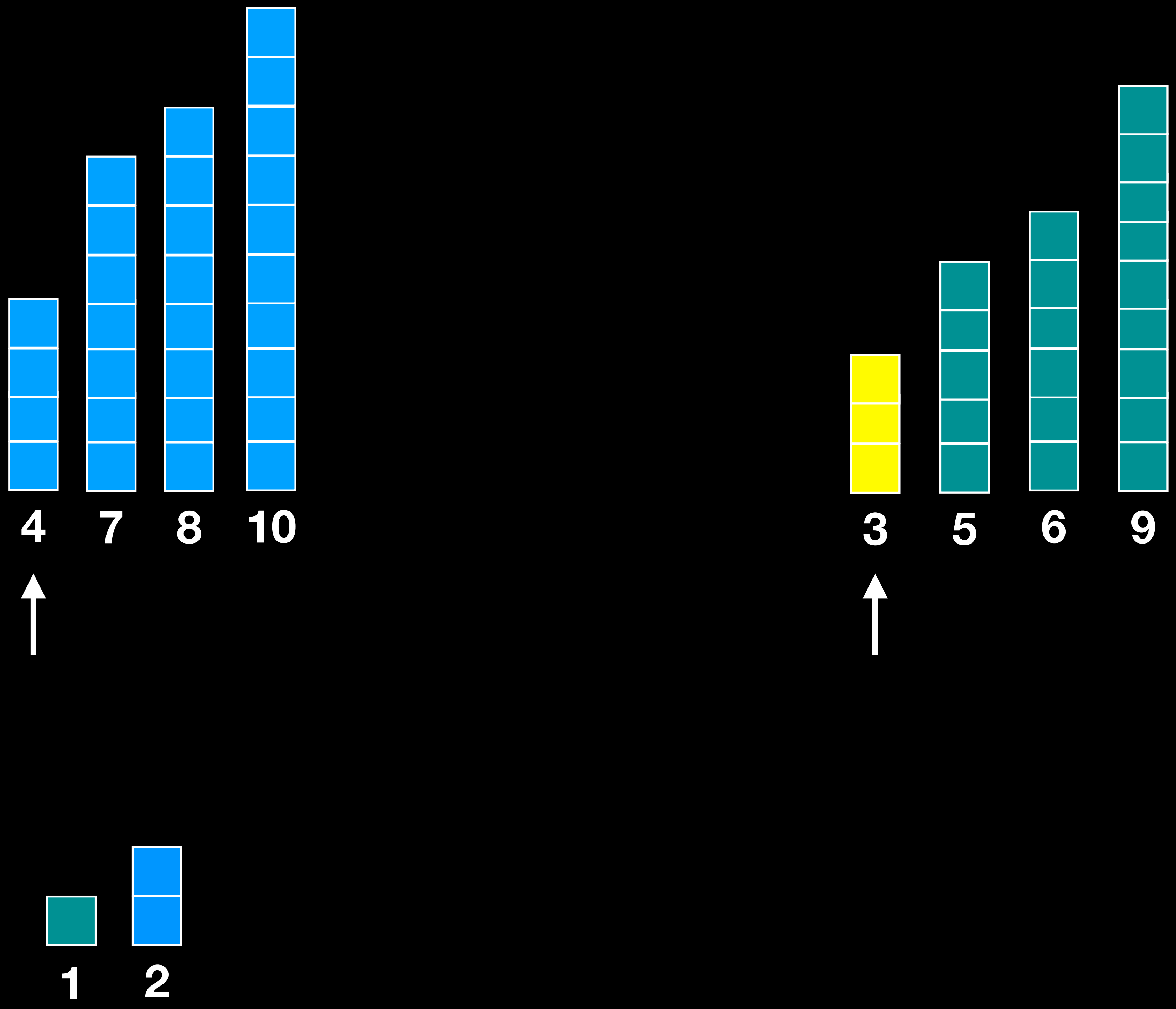
Key Insight: Merge is linear



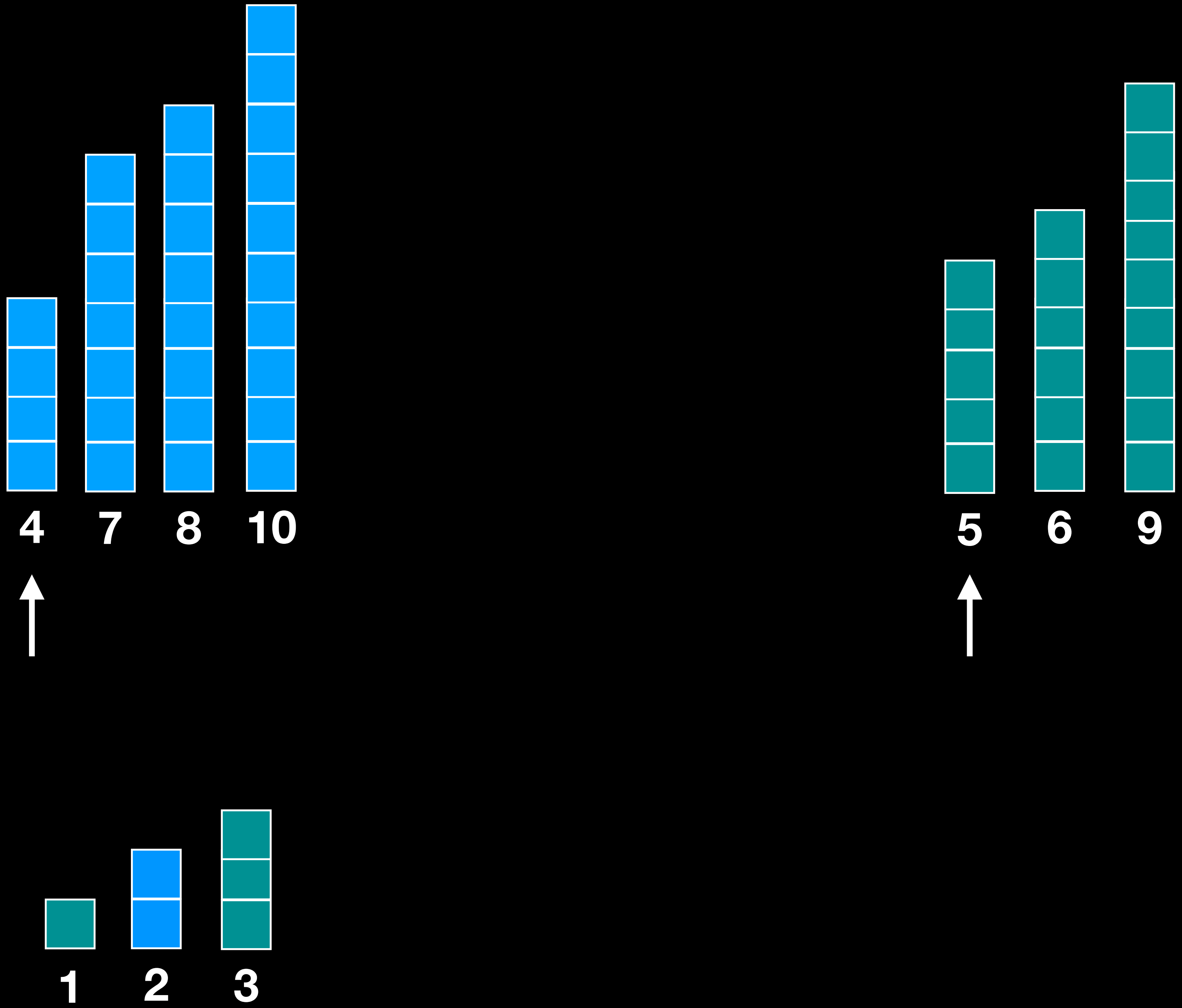
Key Insight: Merge is linear



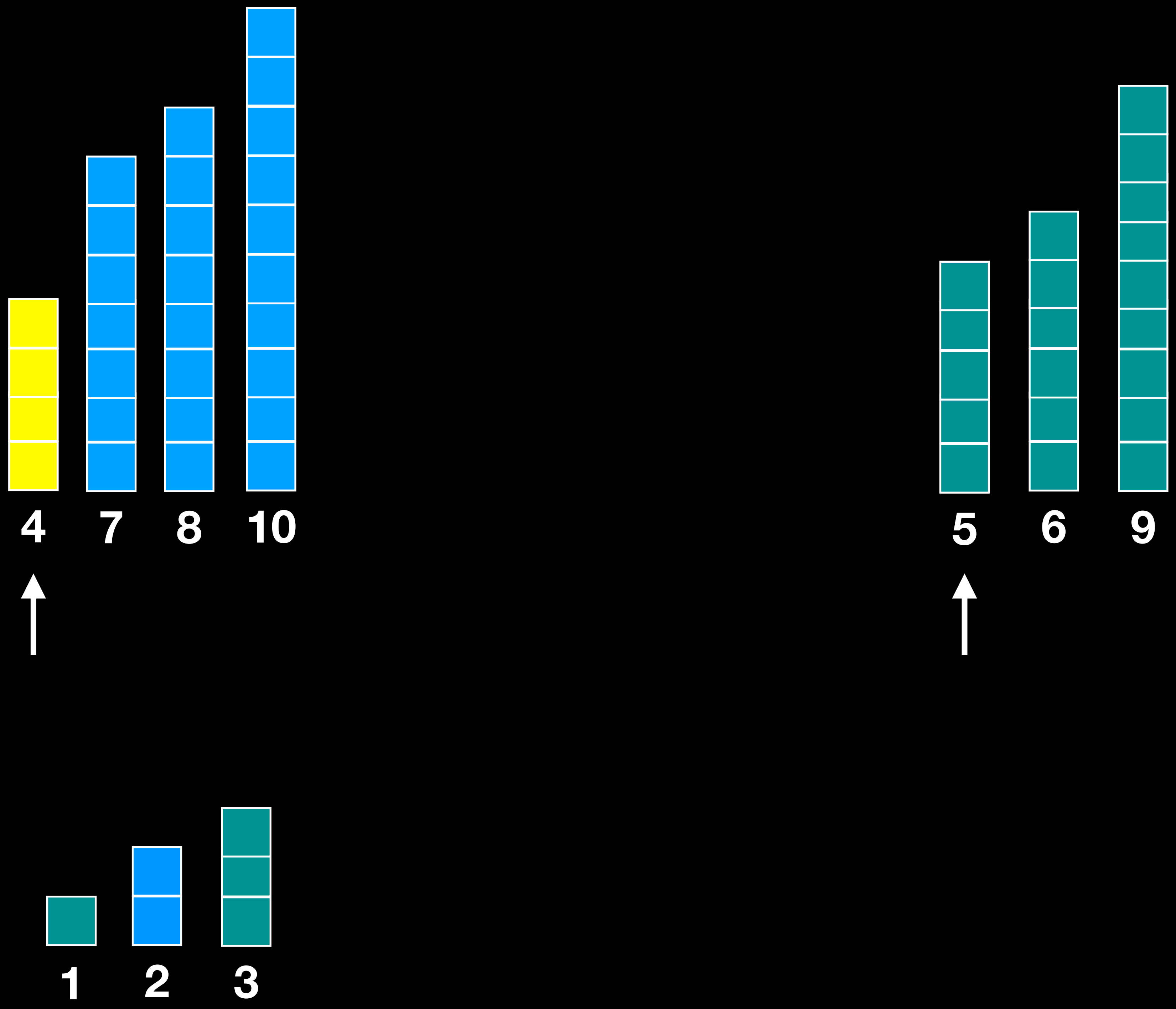
Key Insight: Merge is linear



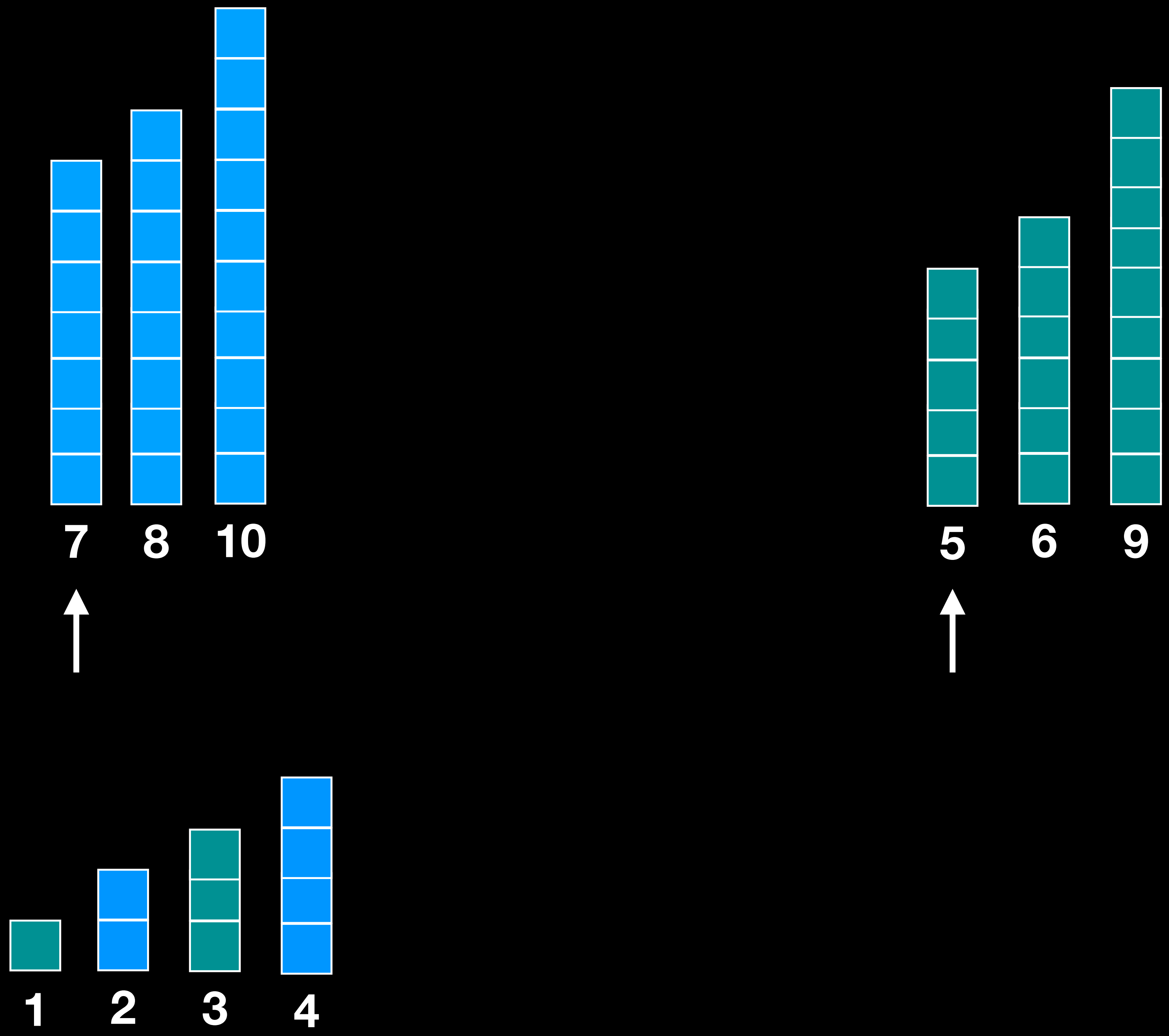
Key Insight: Merge is linear



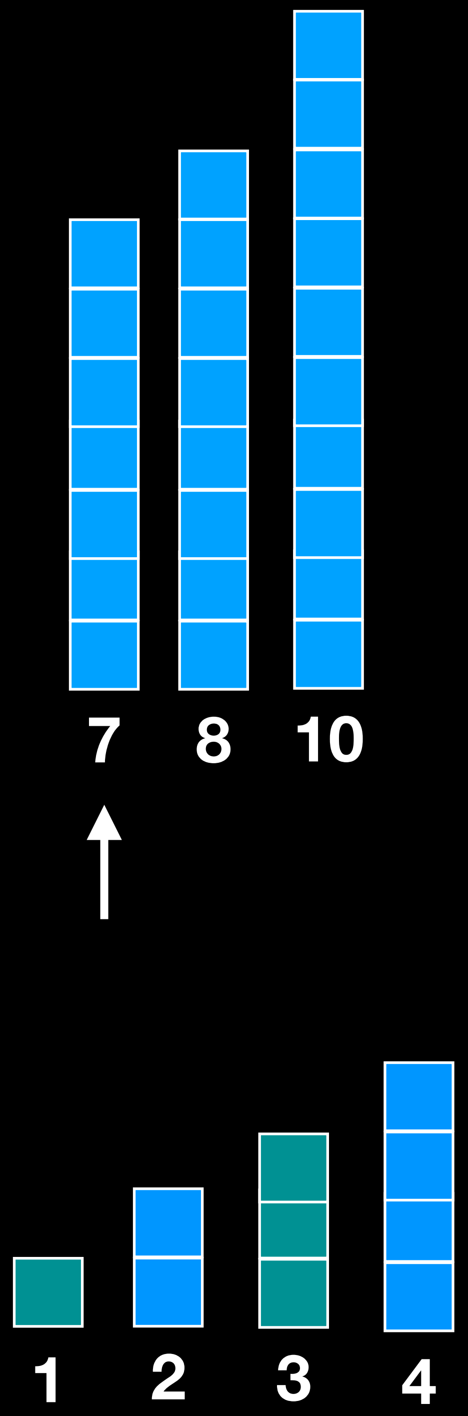
Key Insight: Merge is linear

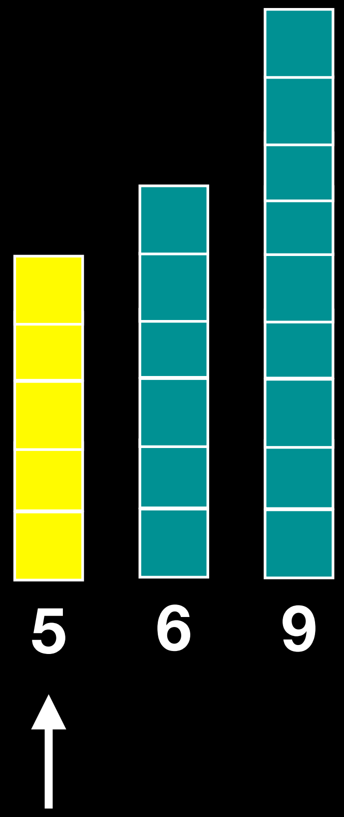


Key Insight: Merge is linear

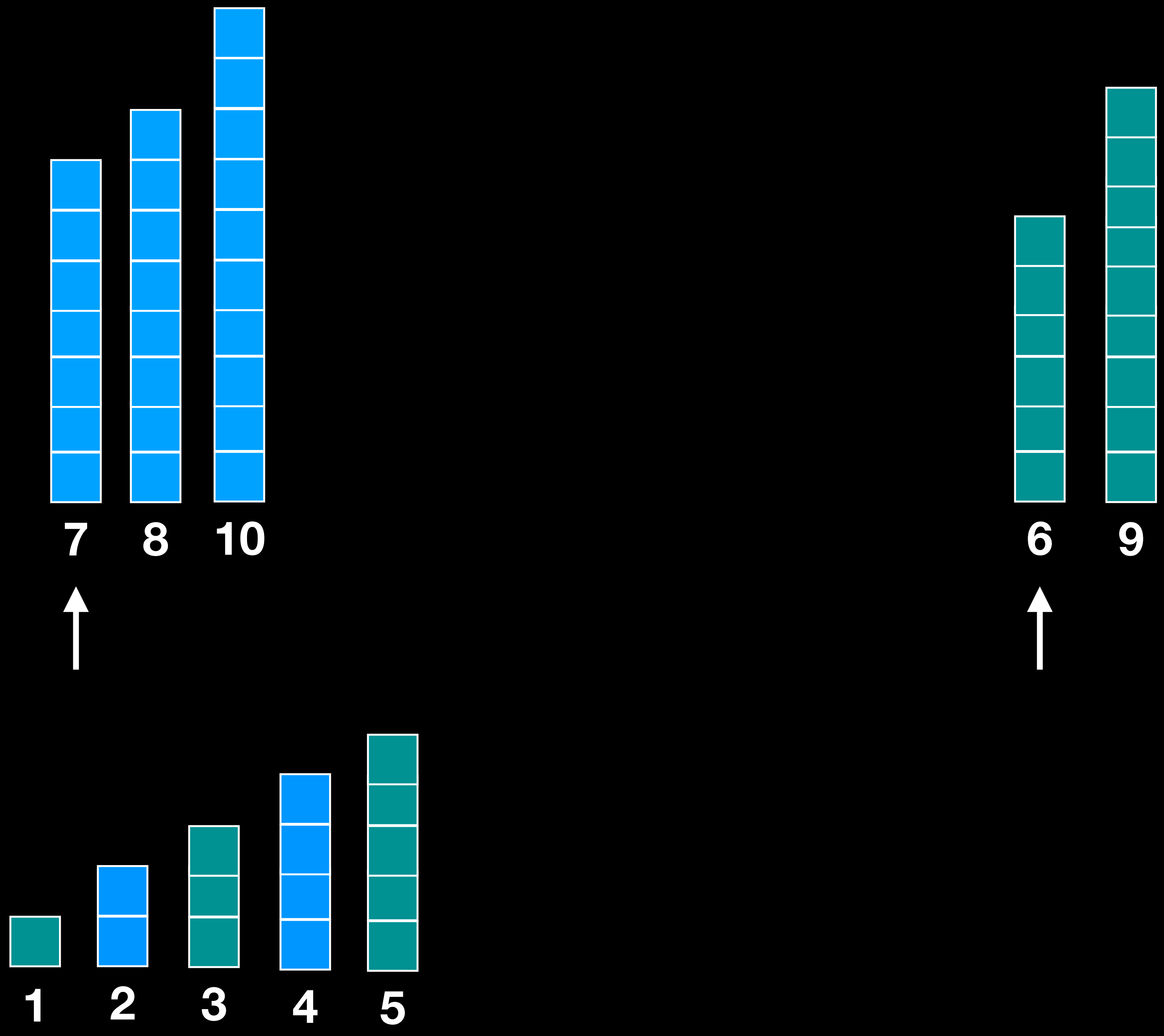


Key Insight: Merge is linear

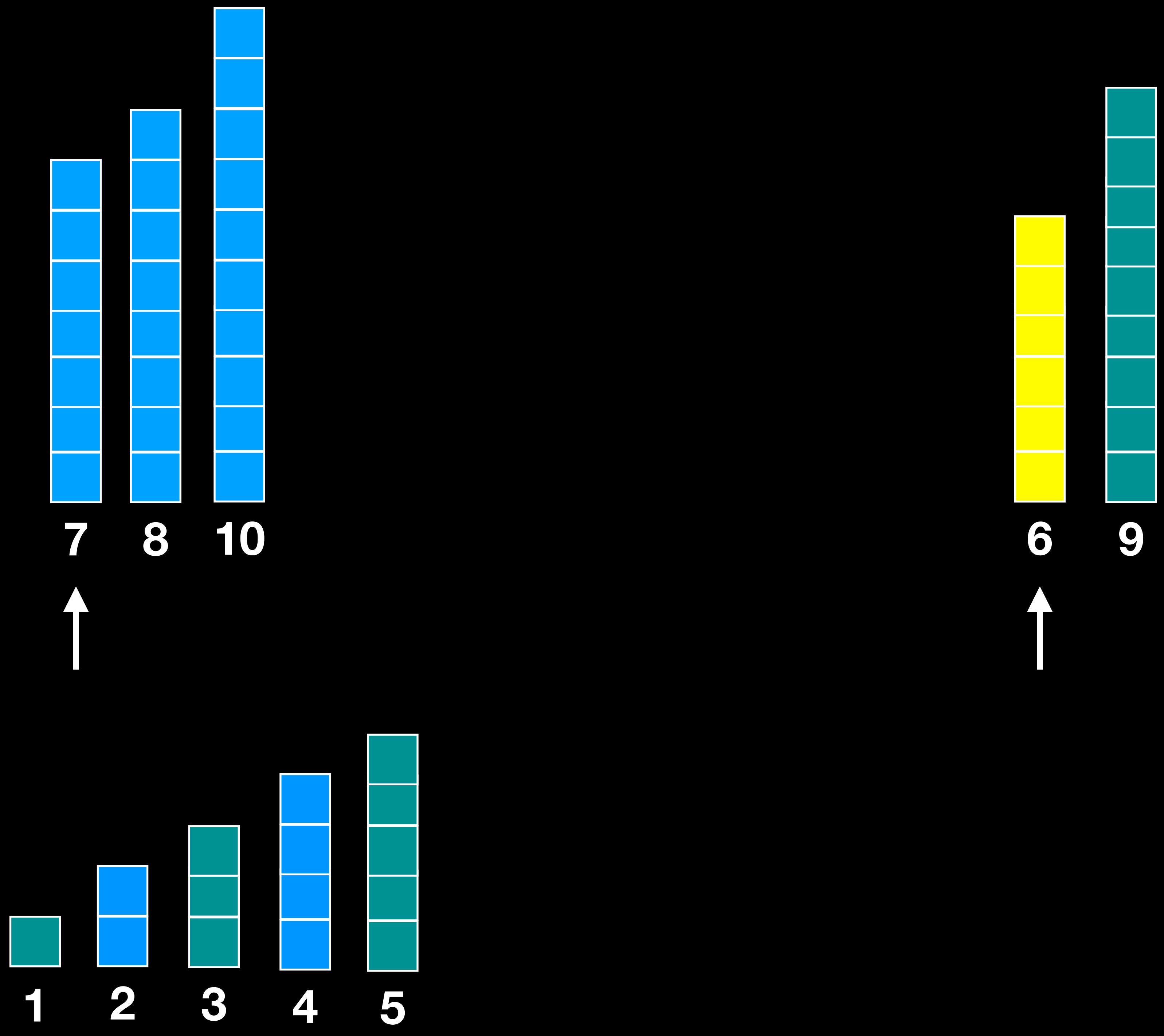




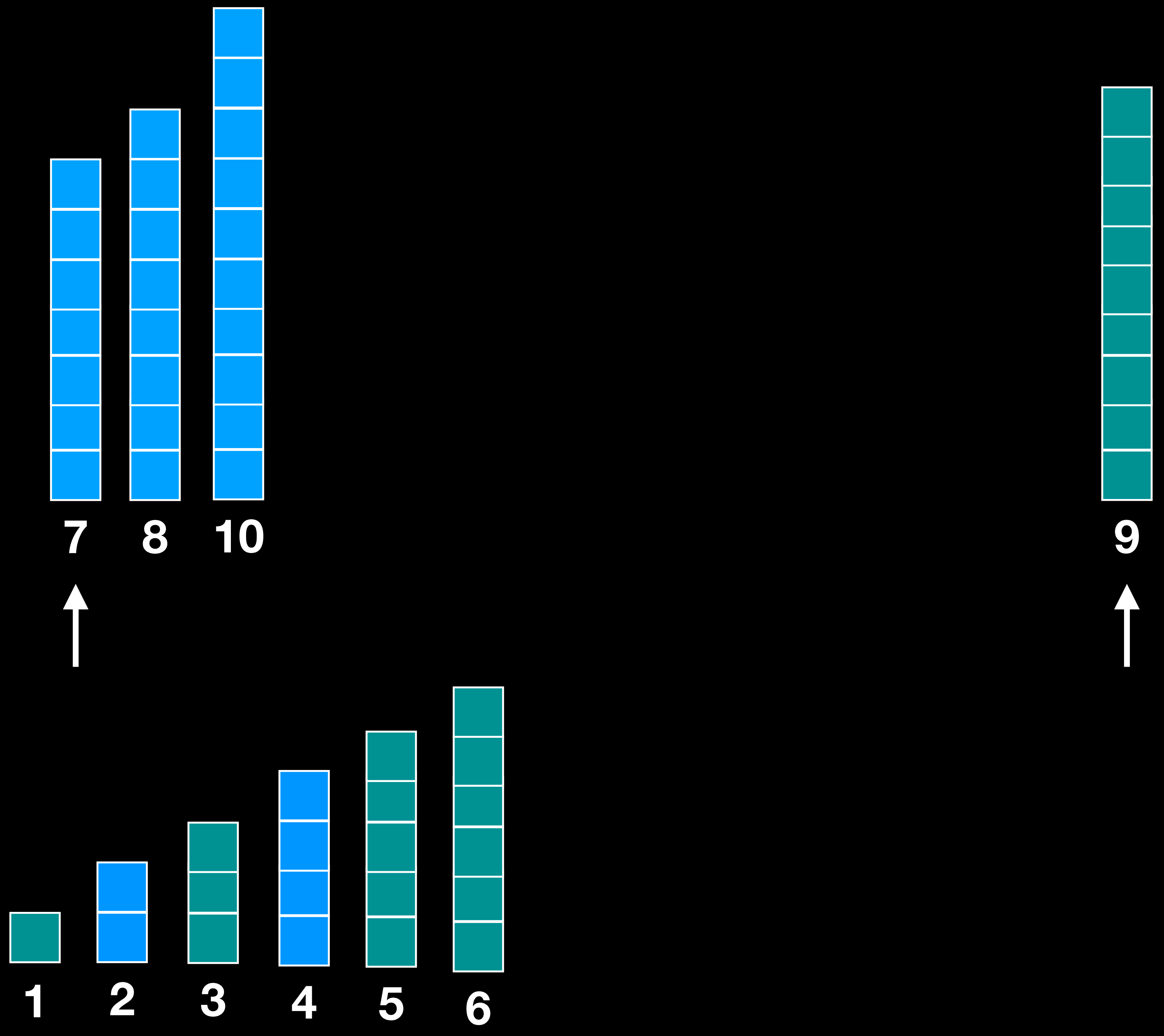
Key Insight: Merge is linear



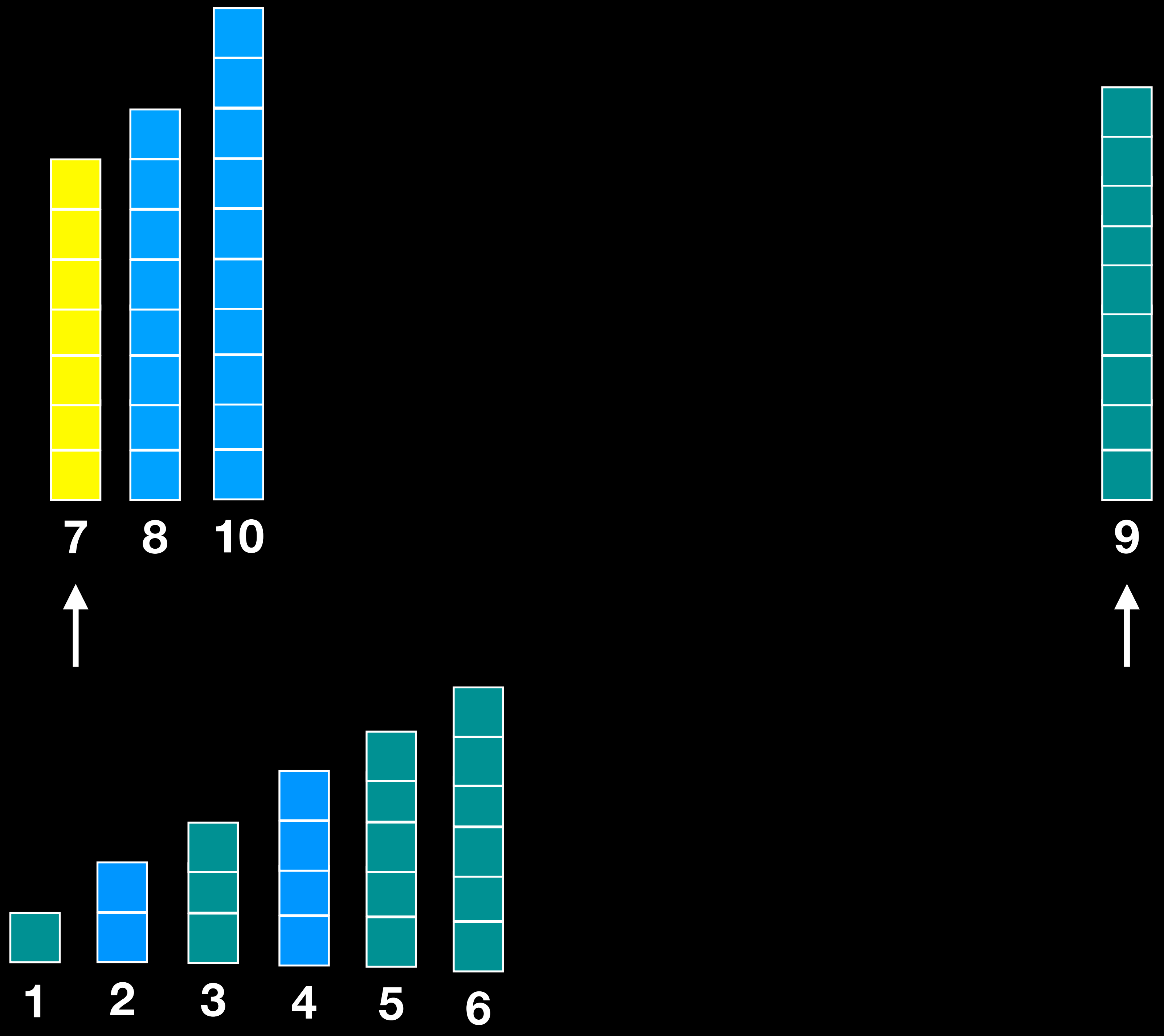
Key Insight: Merge is linear



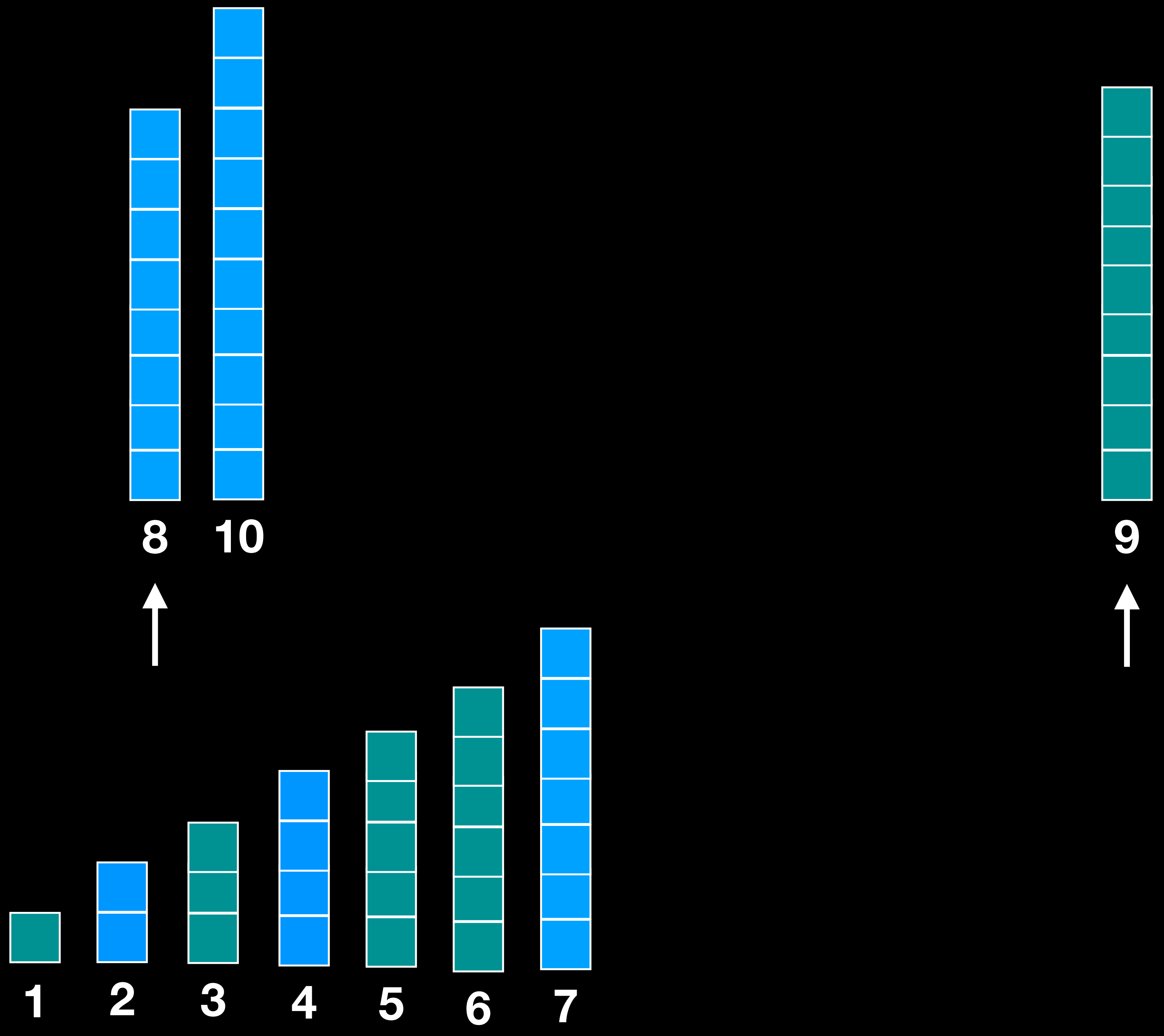
Key Insight: Merge is linear



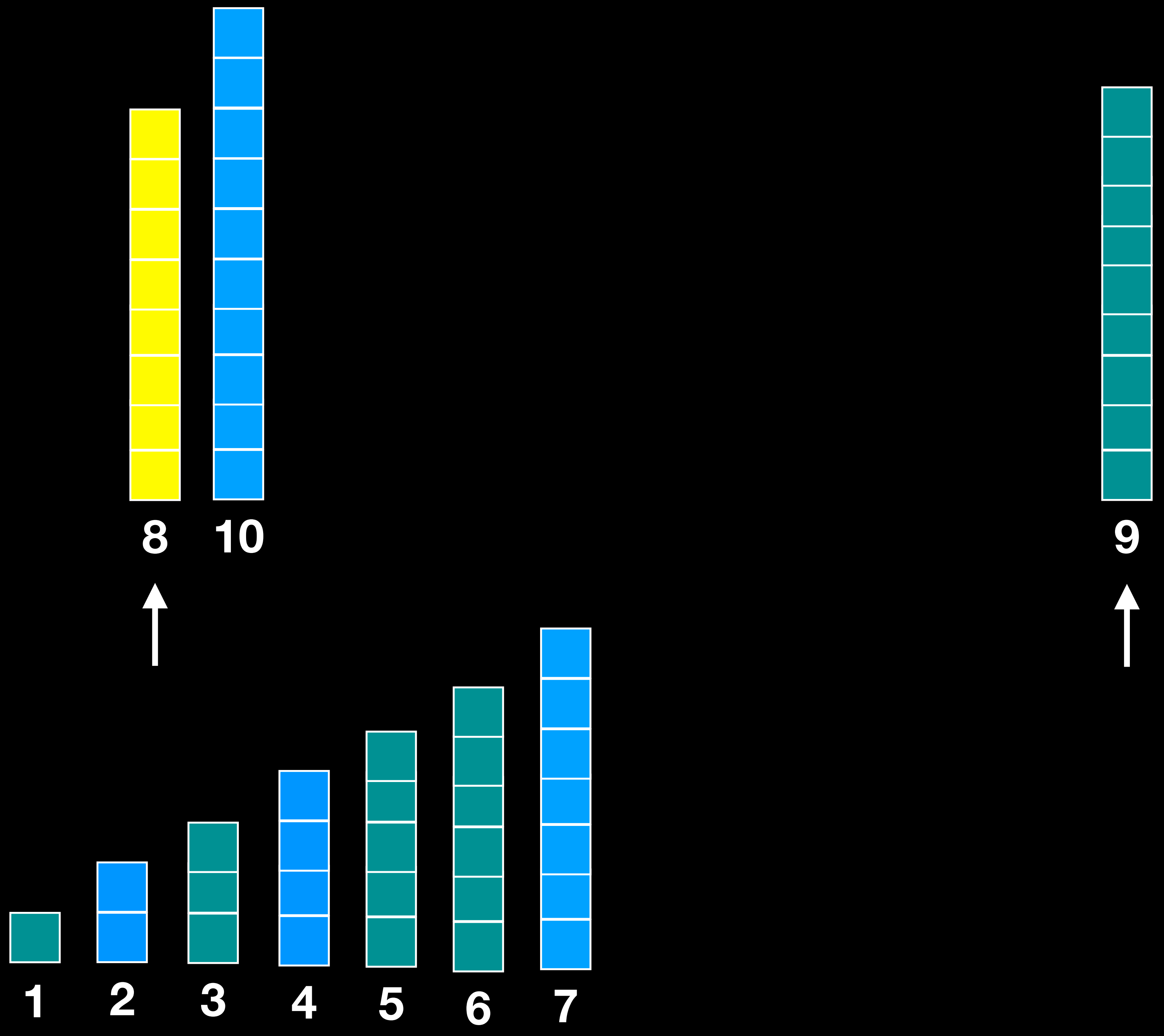
Key Insight: Merge is linear



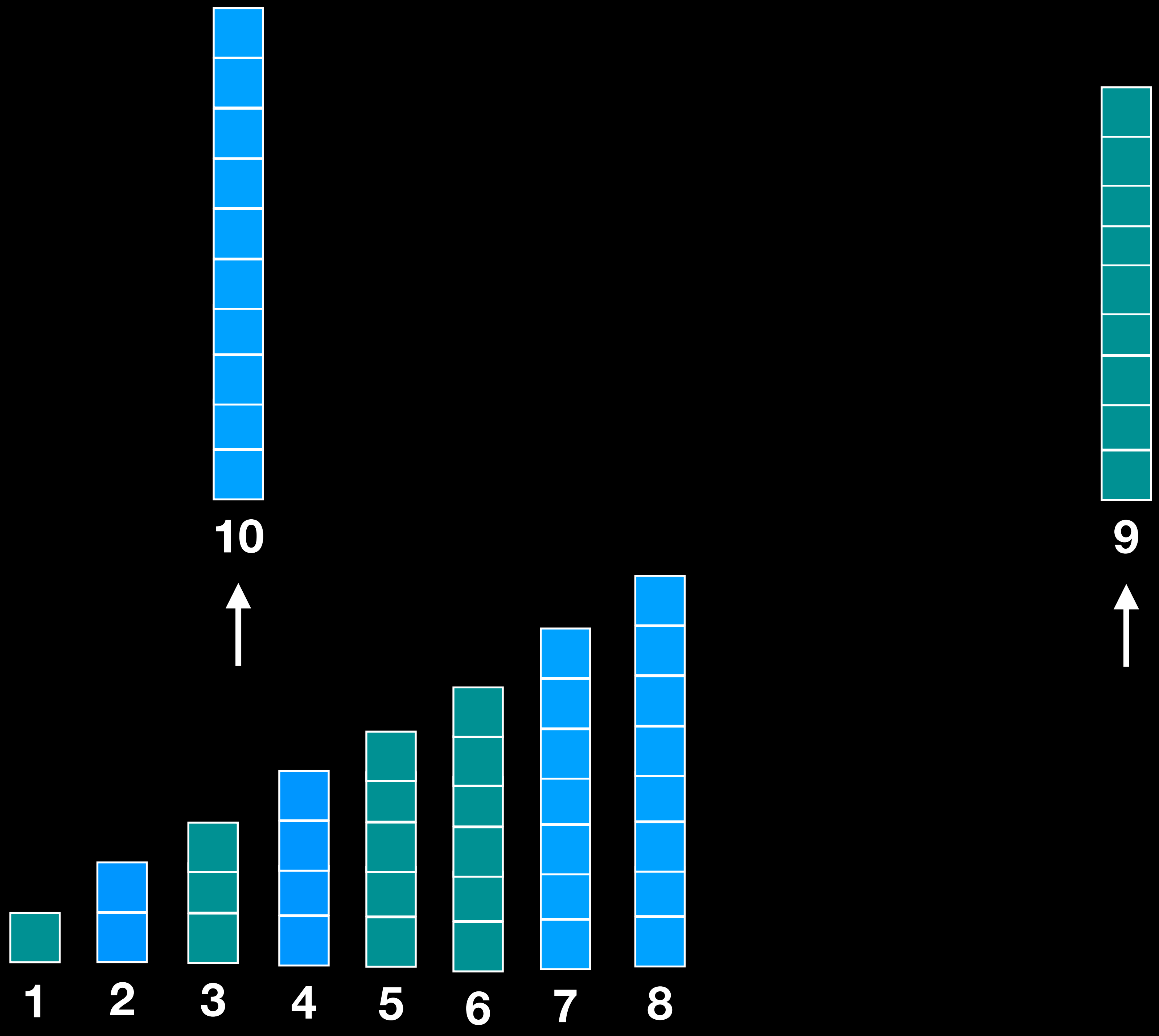
Key Insight: Merge is linear



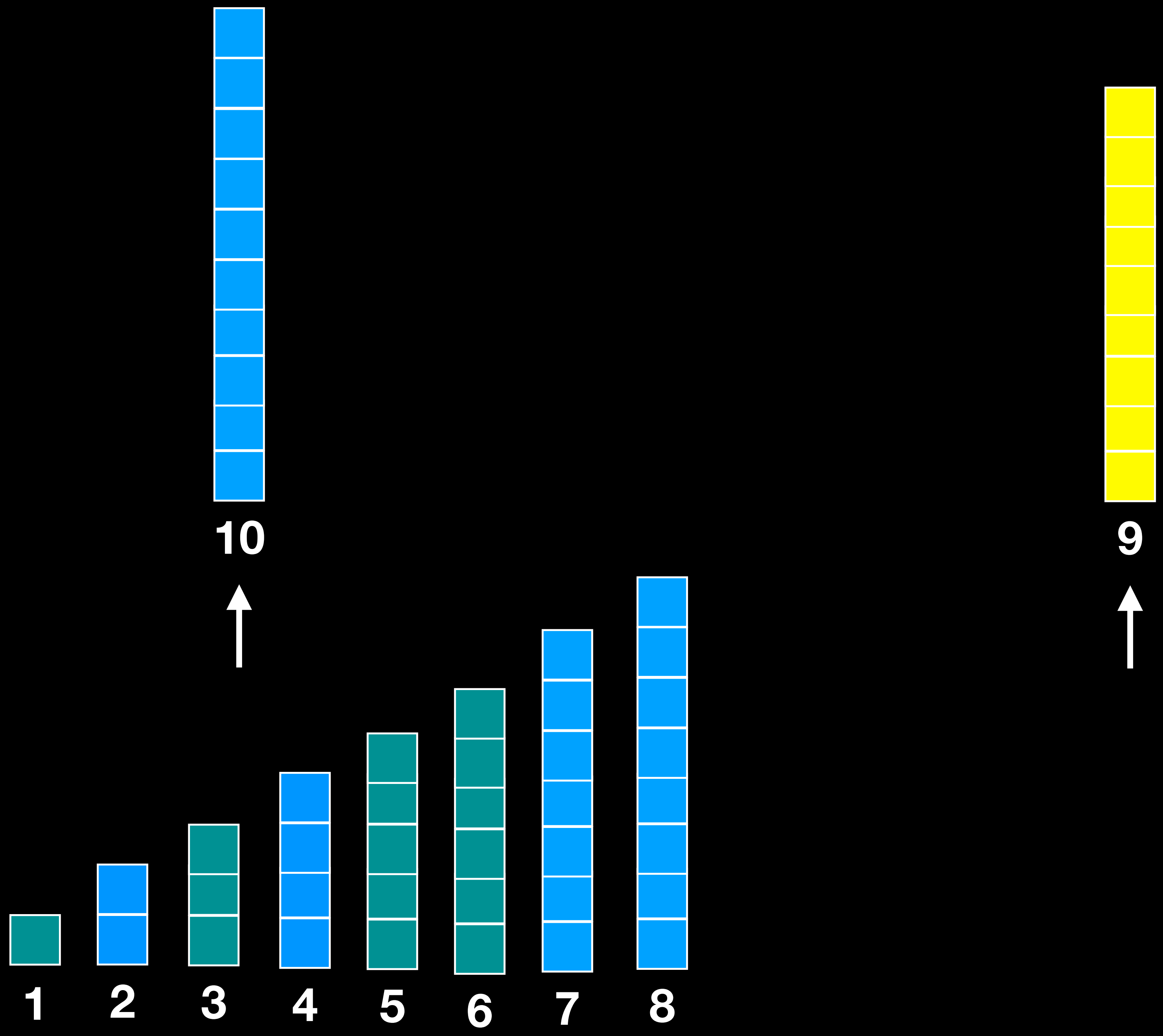
Key Insight: Merge is linear



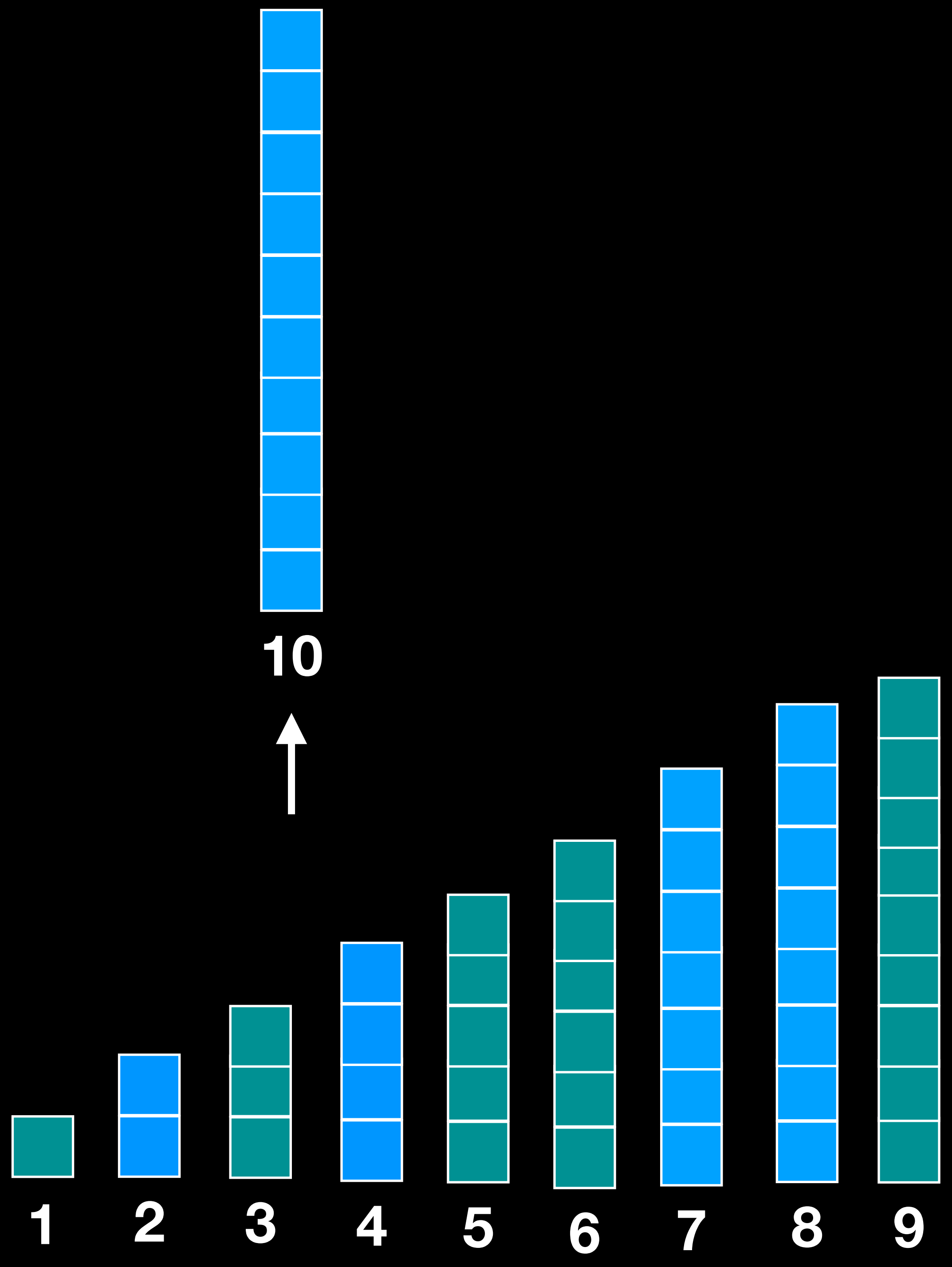
Key Insight: Merge is linear



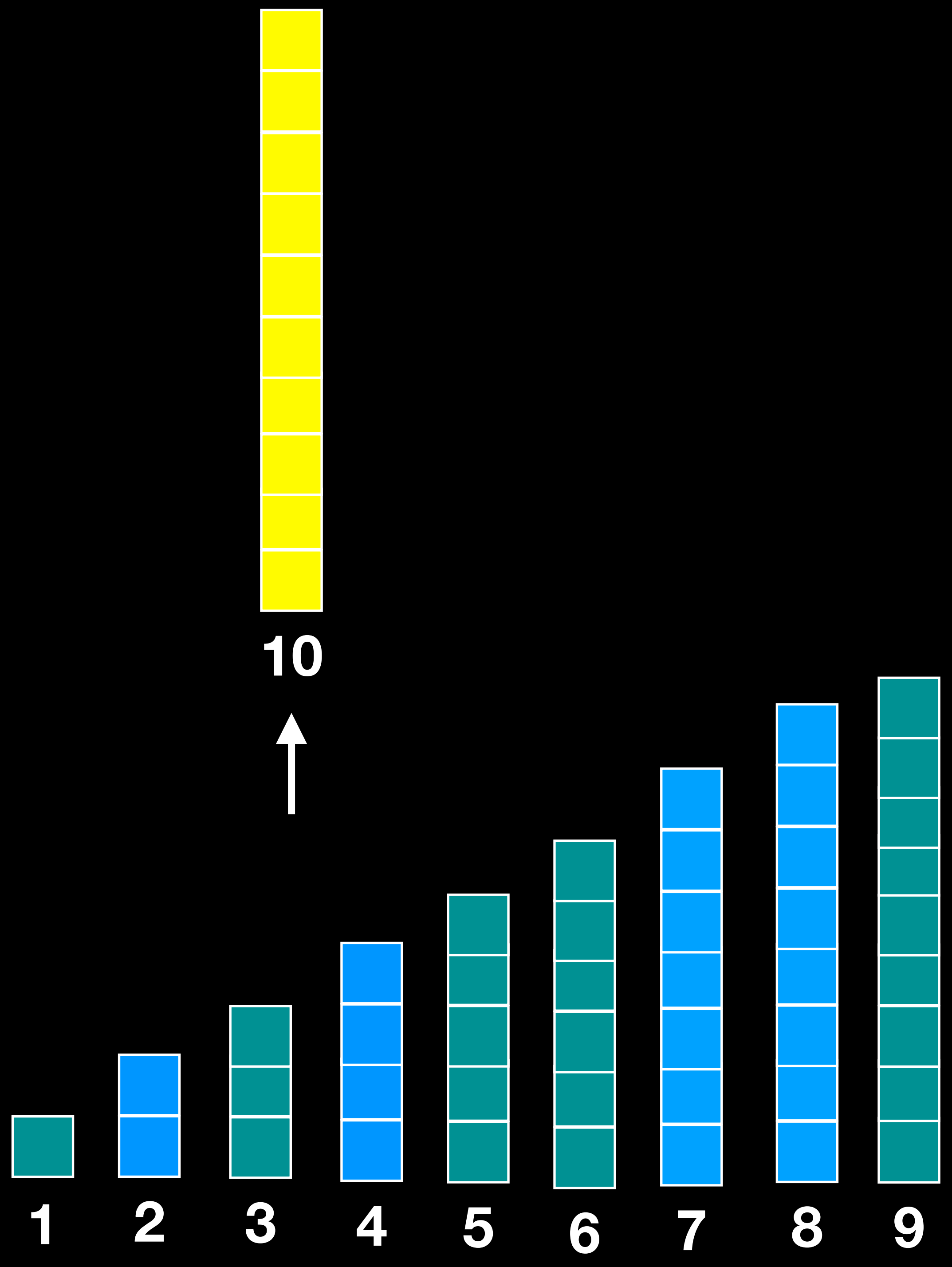
Key Insight: Merge is linear



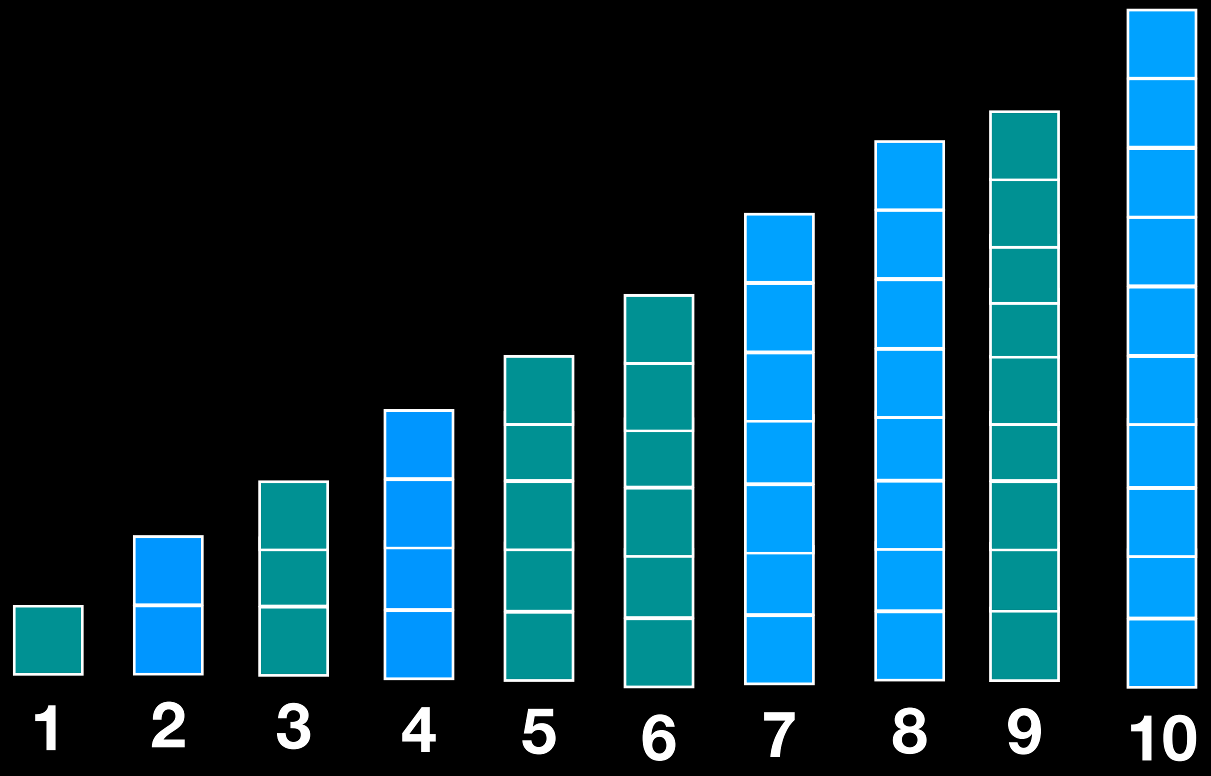
Key Insight: Merge is linear



Key Insight: Merge is linear

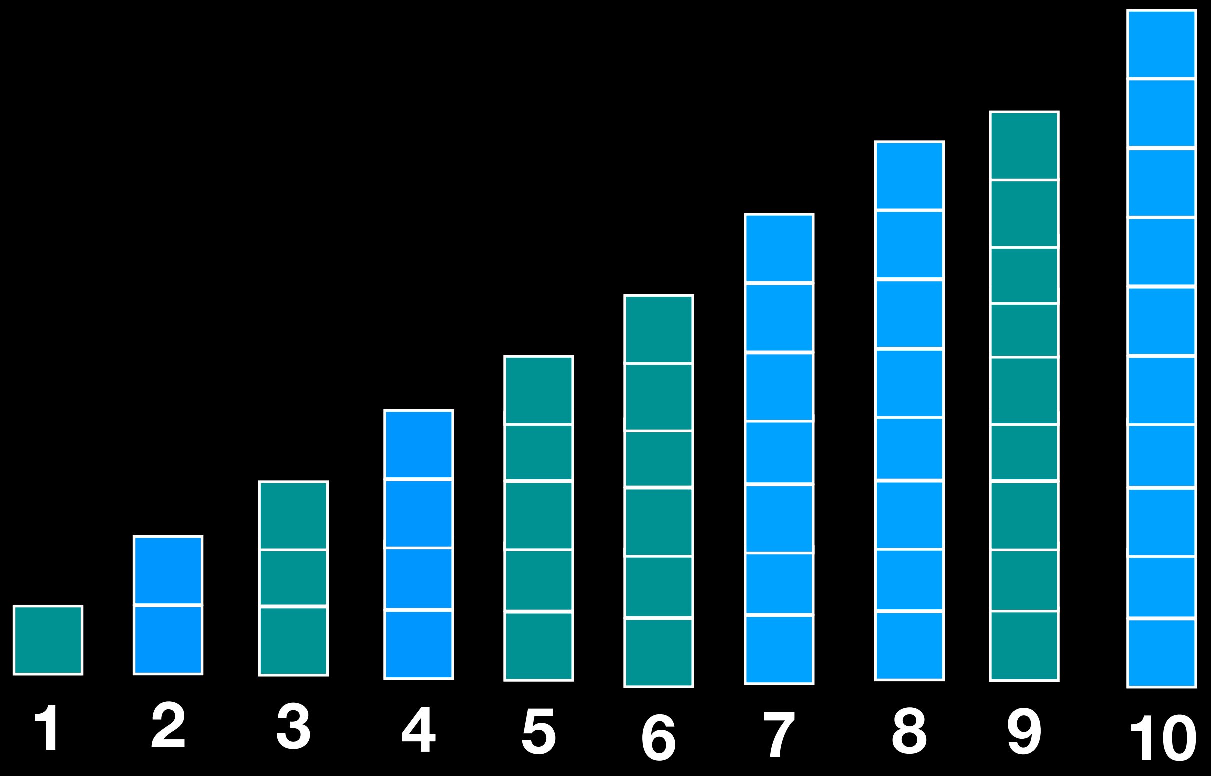


Key Insight: Merge is linear



Key Insight: Merge is linear

Each step makes one comparison and reduces the number of elements to be merged by 1.
If there are *n* total elements to be merged, merging is **O(n)**



Divide and Conquer

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

14	43	76	100	108	200	274	523

11	64	158	195	260	599	932

$T(\frac{1}{2}n) \approx \frac{1}{4} T(n)$

$T(\frac{1}{2}n) \approx \frac{1}{4} T(n)$

Divide and Conquer

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

14	43	76	100	108	200	274	523

11	64	158	195	260	599	932

$T({}^1/_2 n) \approx {}^1/_4\, T(n)$

$T({}^1/_2 n) \approx {}^1/_4\, T(n)$

2	3	5	11	14	43	64	76	100	108	158	195	200	260	274	523	599	932

$T(n) \approx {}^1/_2\, T(n) + n$

Speed up insertion sort by a factor of two by splitting in half, sorting separately and merging results!

Divide and Conquer

Splitting in two gives 2x improvement.

Divide and Conquer

Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Divide and Conquer

Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Splitting in eight gives 8x improvement.

Divide and Conquer

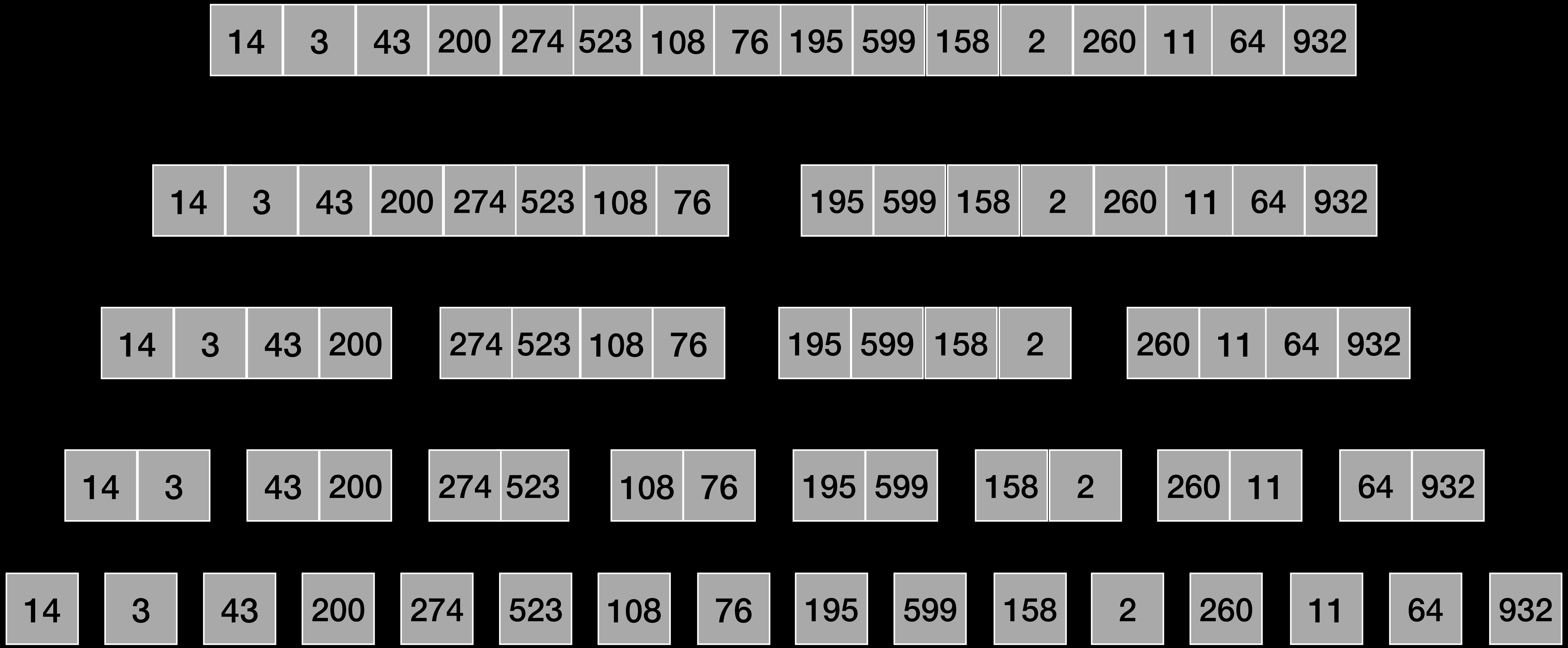
Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Splitting in eight gives 8x improvement.

What if we never stop splitting?

Merge Sort



14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

14 3 43 200 274 523 108 76
195 599 158 2 260 11 64 932

14 3 43 200
274 523 108 76
195 599 158 2
260 11 64 932

14 3
43 200
274 523
108 76
195 599
158 2
260 11
64 932

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523
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599
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260
11
64
932

Merge Sort

14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

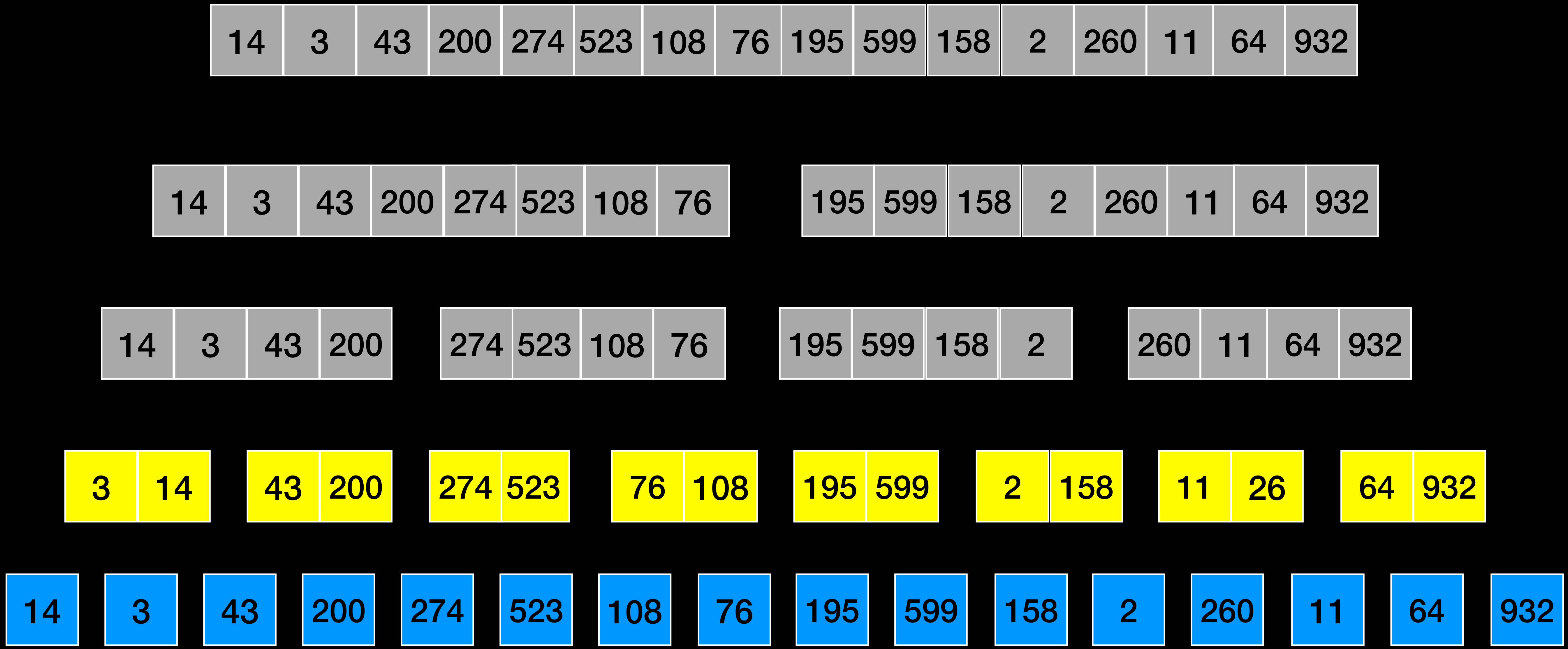
14 3 43 200 274 523 108 76 | 195 599 158 2 260 11 64 932

14 3 43 200 | 274 523 108 76 | 195 599 158 2 | 260 11 64 932

14 3 | 43 200 | 274 523 | 108 76 | 195 599 | 158 2 | 260 11 | 64 932

14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932

Merge Sort



14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

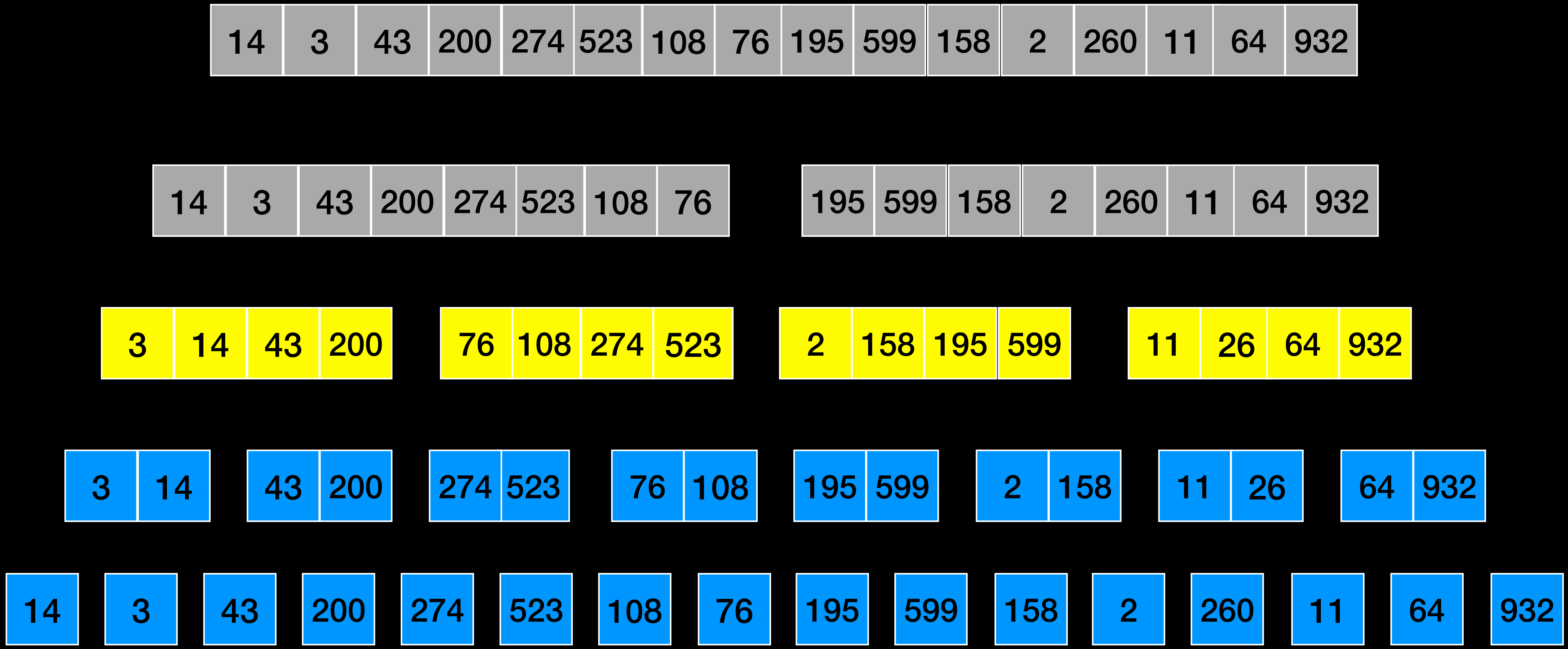
14 3 43 200 274 523 108 76 | 195 599 158 2 260 11 64 932

14 3 43 200 | 274 523 108 76 | 195 599 158 2 | 260 11 64 932

3 14 | 43 200 | 274 523 | 76 108 | 195 599 | 2 158 | 11 26 | 64 932

14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932

Merge Sort



14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

14 3 43 200 274 523 108 76
195 599 158 2 260 11 64 932

3 14 43 200
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2 158 195 599
11 26 64 932

3 14
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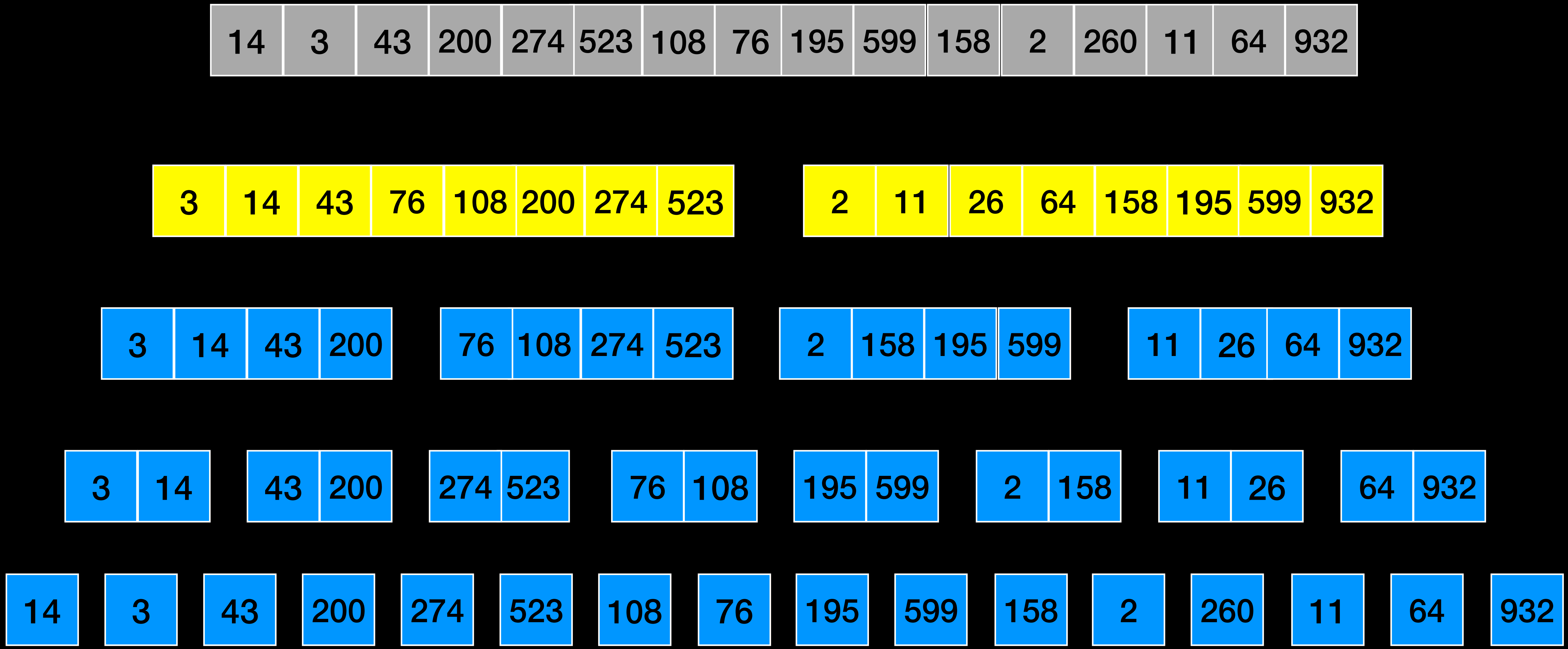
Merge Sort



14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

3 14 43 76 108 200 274 523

2 11 26 64 158 195 599 932

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76 108 274 523

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3 14

43 200

274 523

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2 158

11 26

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260

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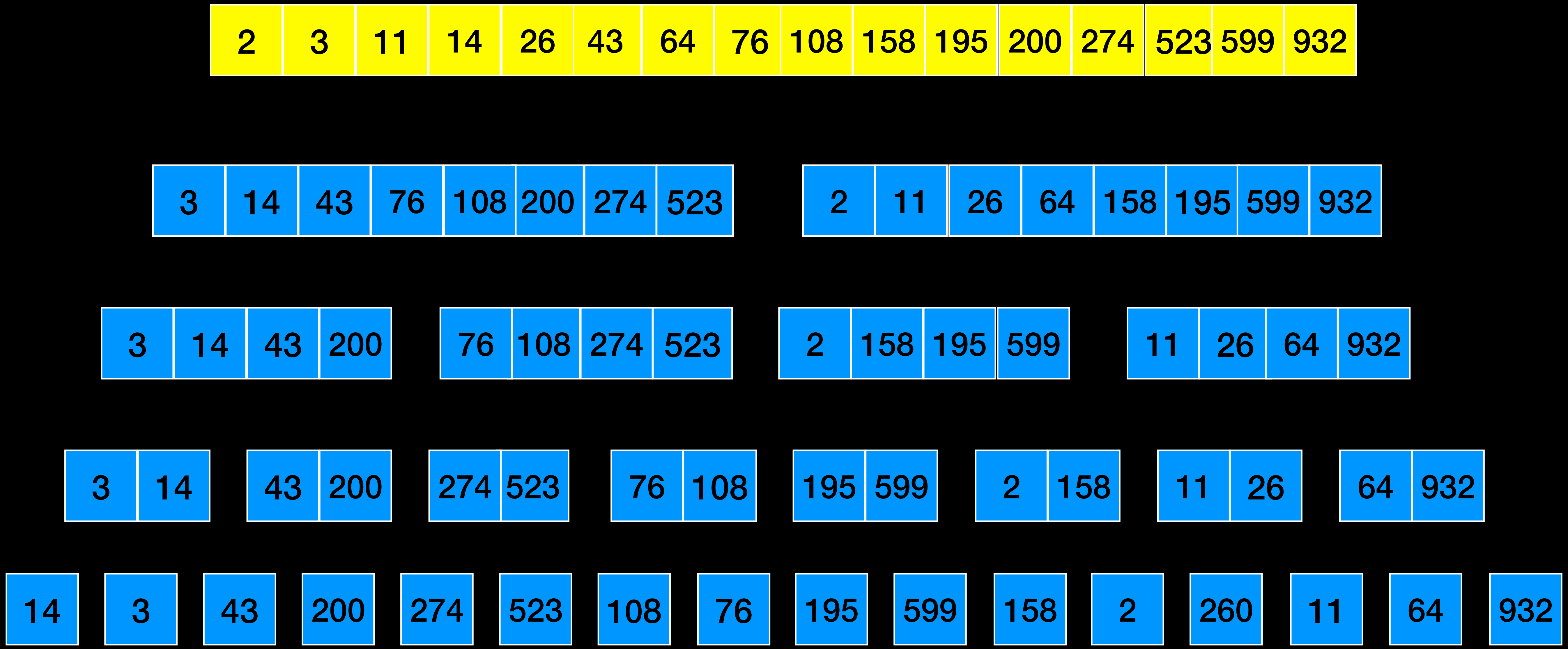
932

Merge Sort



2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932

3 14 43 76 108 200 274 523
2 11 26 64 158 195 599 932

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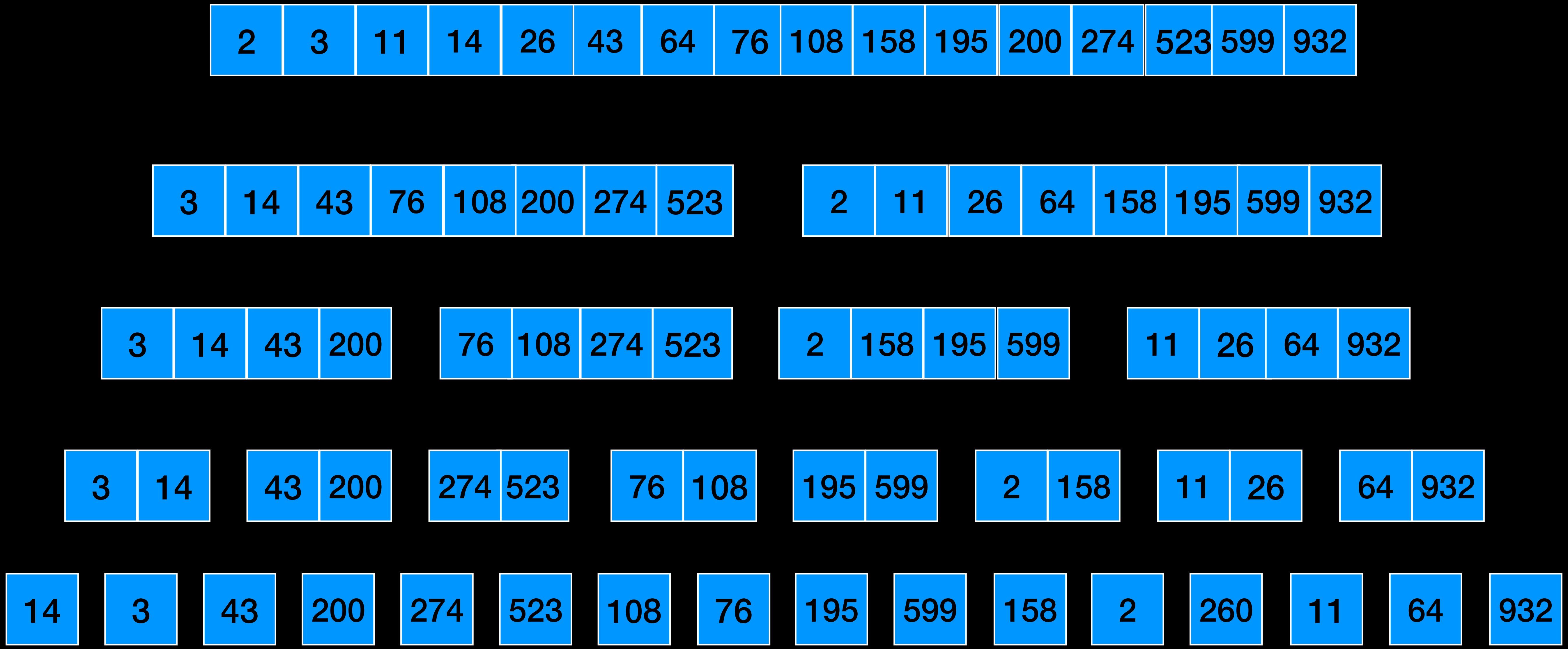
Merge Sort



2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932

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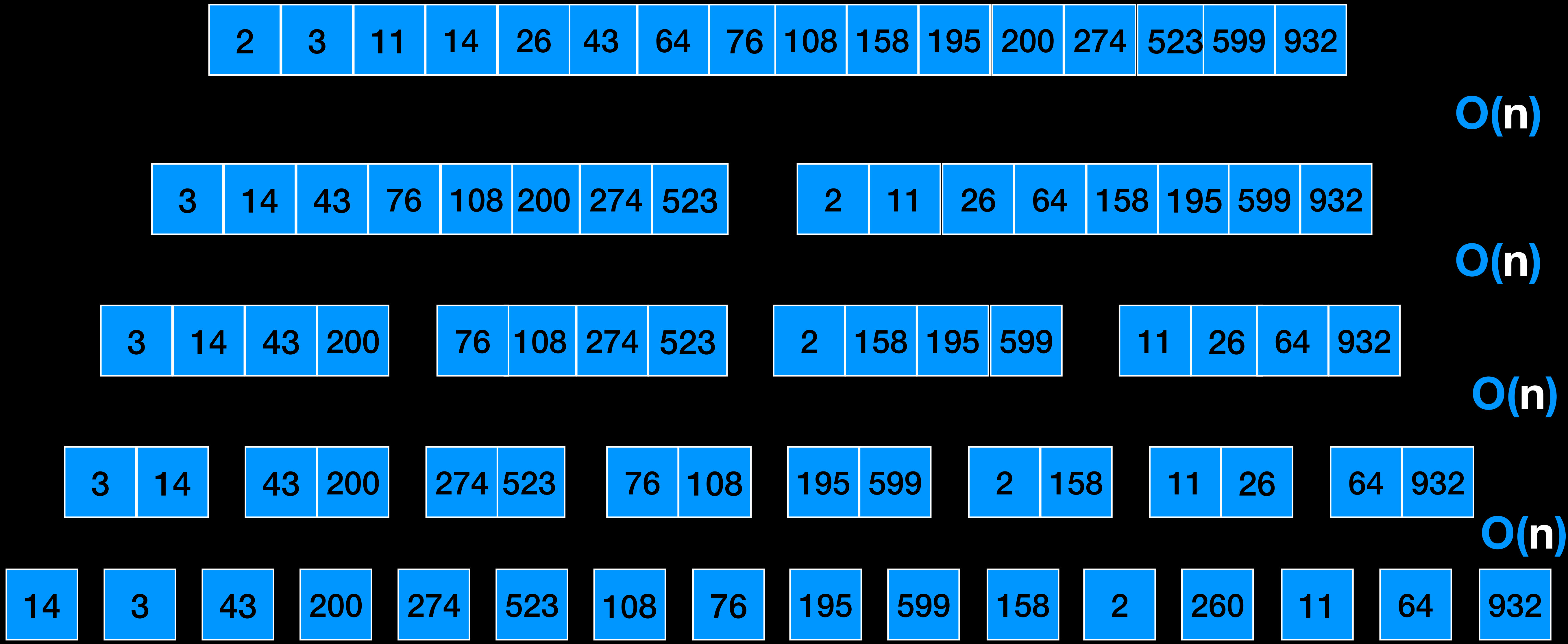
Merge Sort Analysis



2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932

O(n)

3 14 43 76 108 200 274 523

2 11 26 64 158 195 599 932

O(n)

3 14 43 200

76 108 274 523

2 158 195 599

11 26 64 932

O(n)

3 14

43 200

274 523

76 108

195 599

2 158

11 26

64 932

O(n)

14

3

43

200

274

523

108

76

195

599

158

2

260

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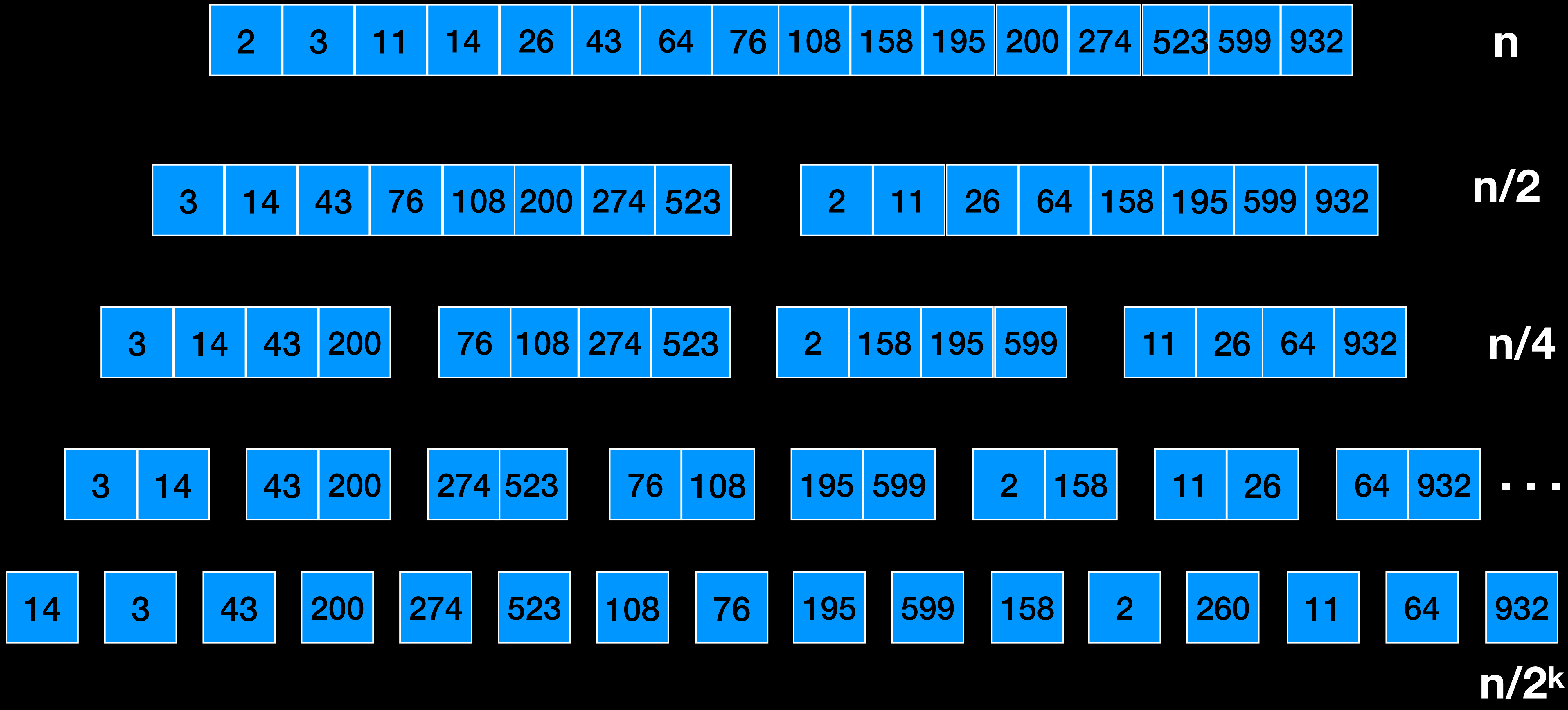
64

932

Merge Sort Analysis



2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932 **n**

3 14 43 76 108 200 274 523   2 11 26 64 158 195 599 932 **n/2**

3 14 43 200   76 108 274 523   2 158 195 599   11 26 64 932 **n/4**

3 14   43 200   274 523   76 108   195 599   2 158   11 26   64 932 . . .

14   3   43   200   274   523   108   76   195   599   158   2   260   11   64   932

$n/2^k$

Merge n how many times?

Merge Sort Analysis

2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932 — n

3 14 43 76 108 200 274 523 | 2 11 26 64 158 195 599 932 — n/2

3 14 43 200 | 76 108 274 523 | 2 158 195 599 | 11 26 64 932 — n/4

3 14 | 43 200 | 274 523 | 76 108 | 195 599 | 2 158 | 11 26 | 64 932 . . .

14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932

$n/2^k$

Merge n how may times? $n/2^k = 1$

$n = 2^k$

$\log_2 n = k$

Merge Sort Analysis

2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932 — n

3 14 43 76 108 200 274 523 | 2 11 26 64 158 195 599 932 — n/2

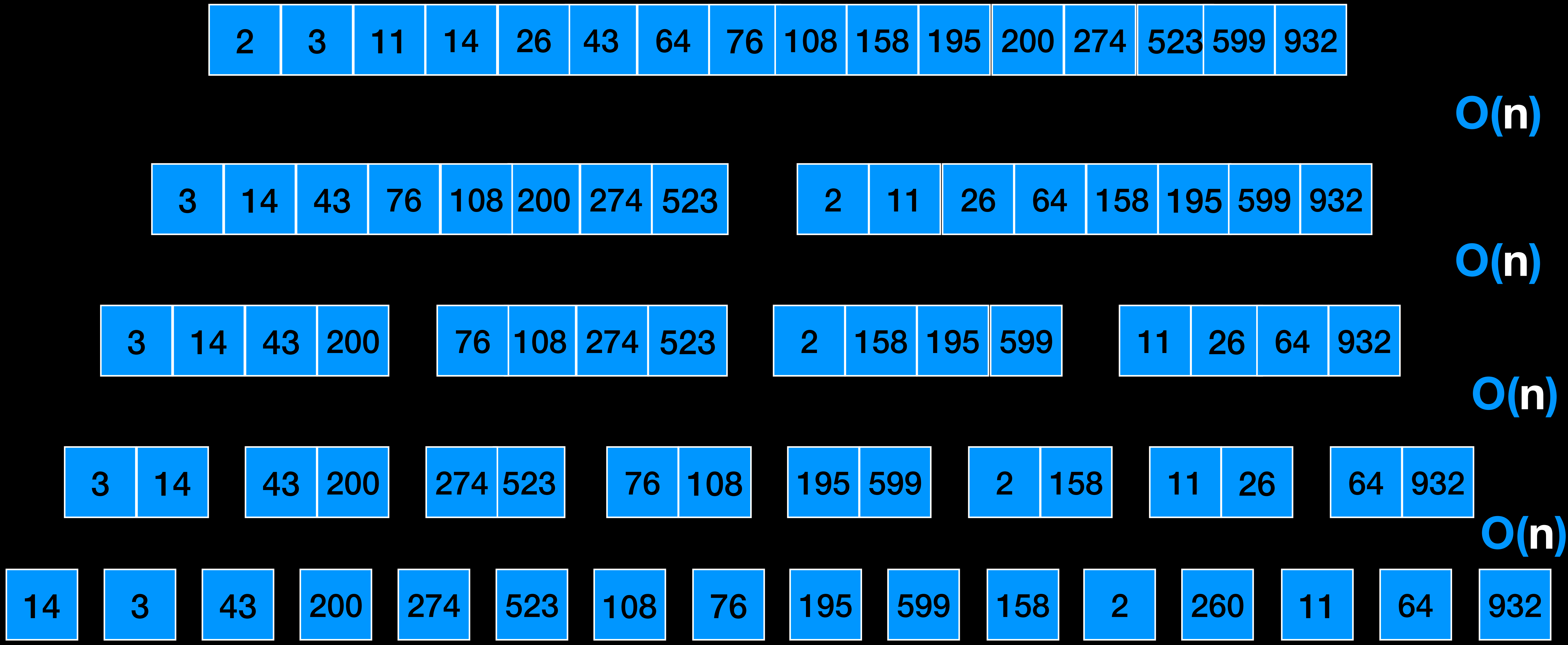
3 14 43 200 | 76 108 274 523 | 2 158 195 599 | 11 26 64 932 — n/4

3 14 | 43 200 | 274 523 | 76 108 | 195 599 | 2 158 | 11 26 | 64 932 . . .

14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932 — $n/2^k$

Merge n elements $\log_2 n$ times

Merge Sort Analysis



2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932

O(n)

3 14 43 76 108 200 274 523

2 11 26 64 158 195 599 932

O(n)

3 14 43 200

76 108 274 523

2 158 195 599

11 26 64 932

O(n)

3 14

43 200

274 523

76 108

195 599

2 158

11 26

64 932

O(n)

14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

O(n log n)

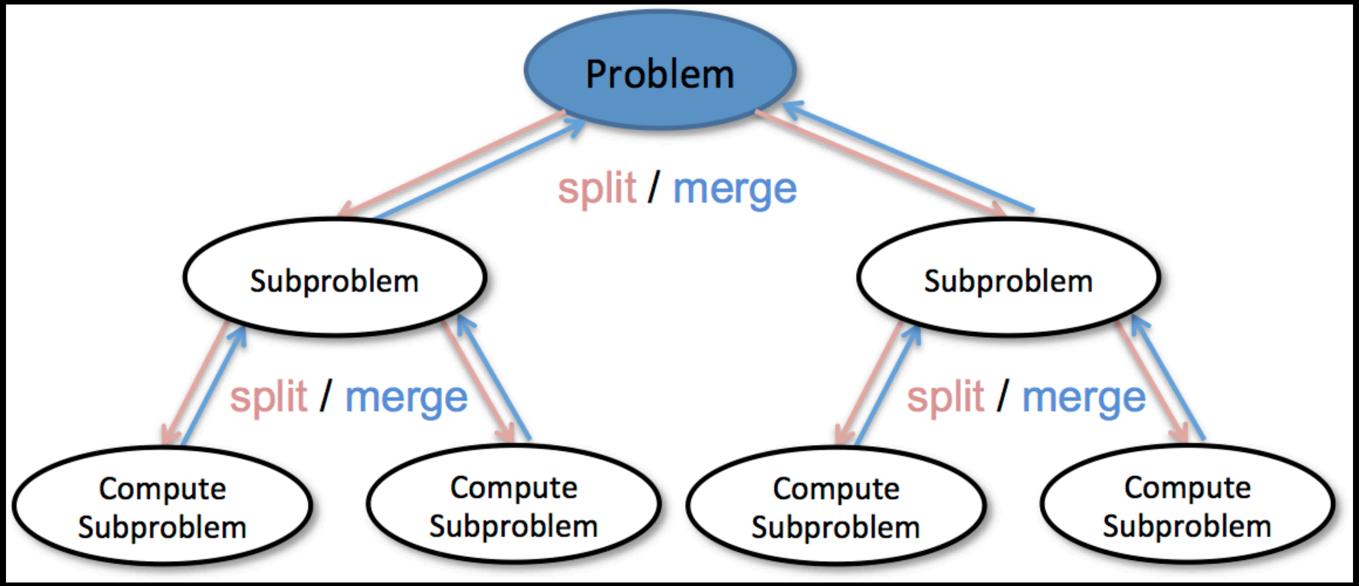
Merge Sort

How would you code this?

Merge Sort

How would you code this?

Hint: Divide and Conquer!!!



Merge Sort

```
void mergeSort(array)
{
    if array size <= 1
        return   //base case
    split array into left_array and right_array
  → mergeSort(left_array)
  → mergeSort(right_array)

    merge(left_array, right_array, array)

}
```

Now sorted: contains left and right merged

Merge Sort Analysis

Execution time does NOT depend on initial arrangement of data

Worst Case: O(n log n) comparisons and data moves

Best Case: O(n log n) comparisons and data moves

Stable

Best we can do with *comparison-based* sorting that does not rely on a data structure in the worst case => can't beat O(n log n)

Space overhead: auxiliary array at each merge step

What we have so far

	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n^2)$	$O(n)$
Bubble Sort	$O(n^2)$	$O(n)$
Merge Sort	$O(n \log n)$	$O(n \log n)$

Quick Sort

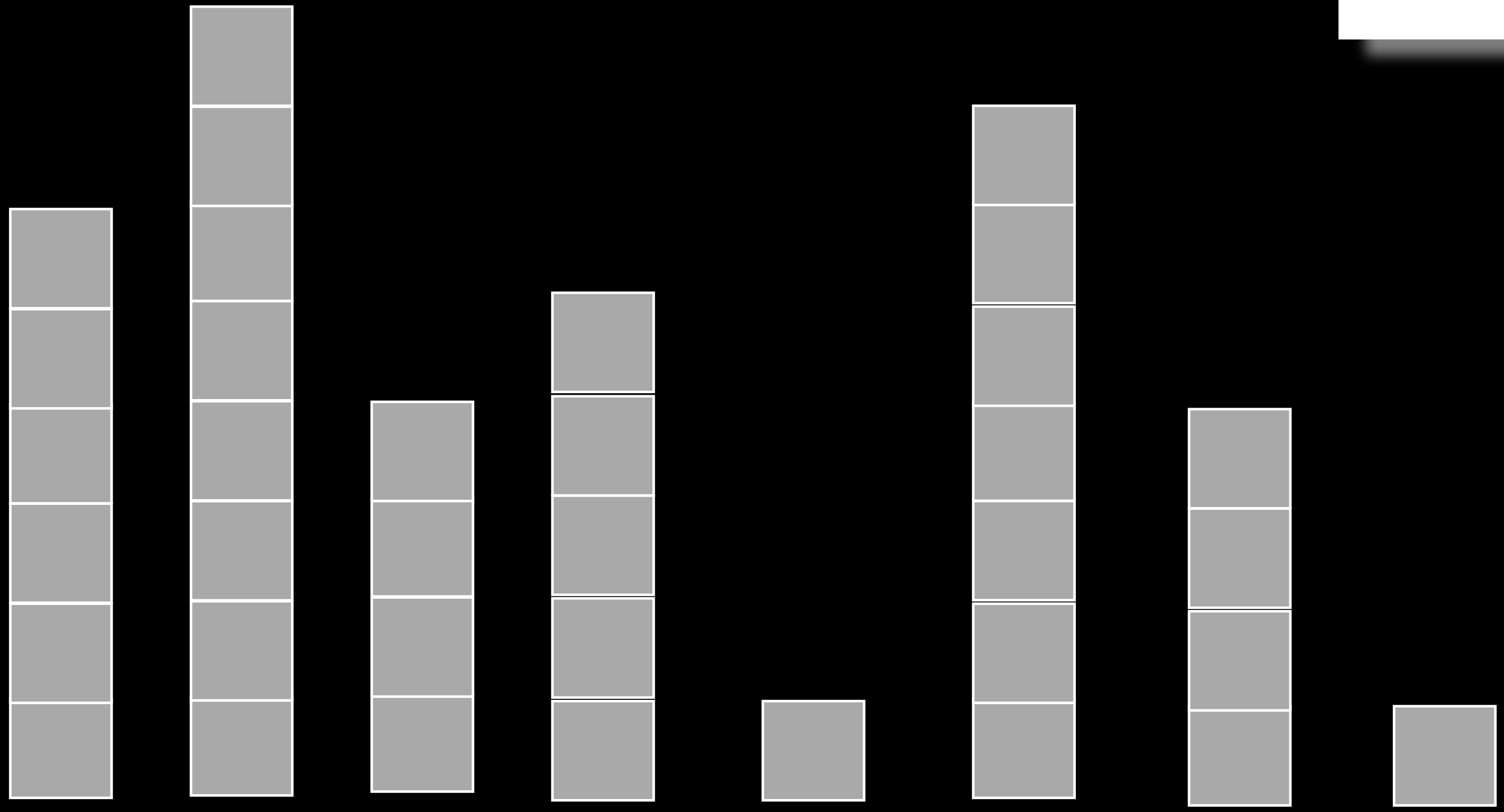
Quick Sort

⬜ <= pivot

▨ > pivot

Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

Partition

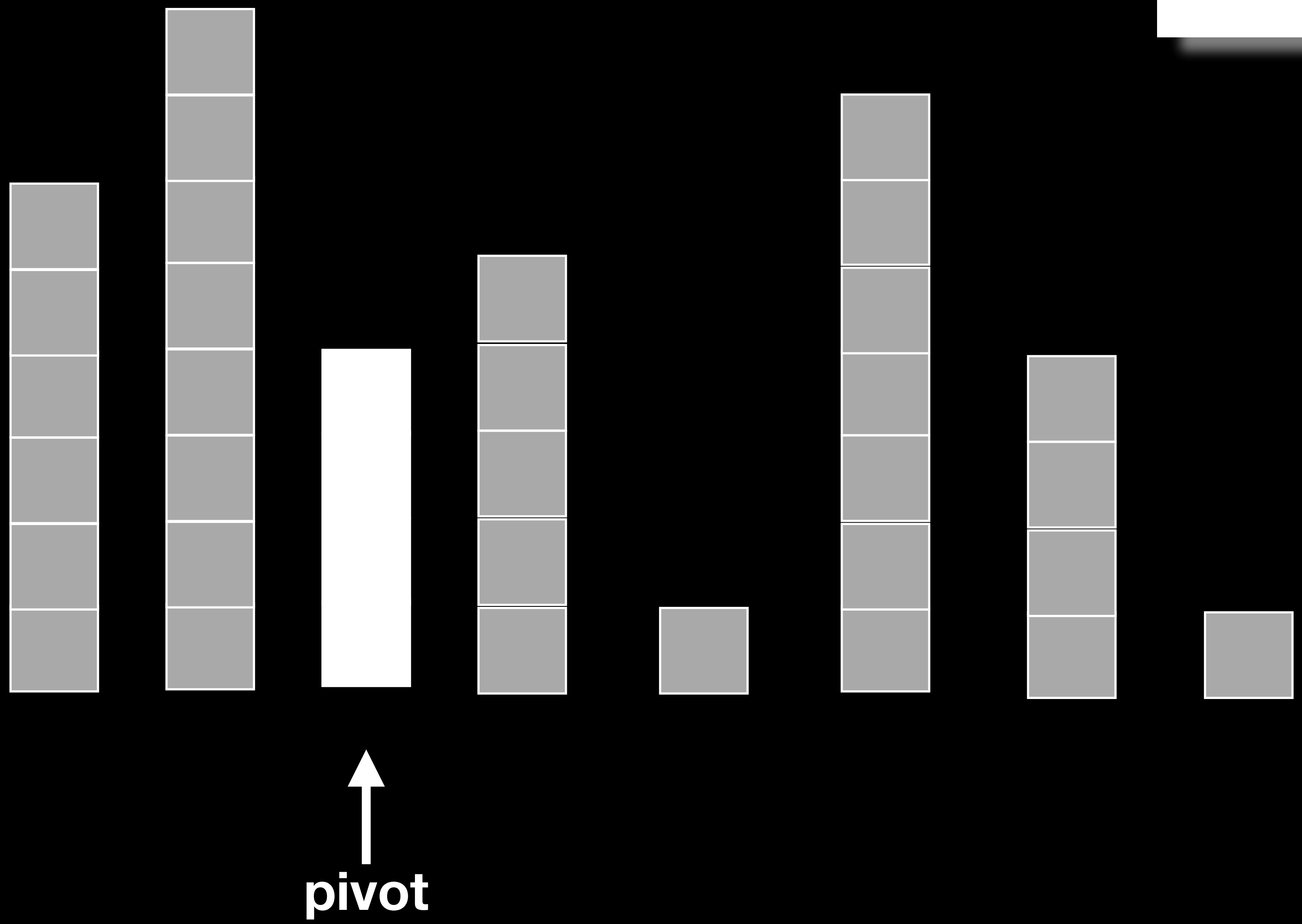


Quick Sort

⬜ <= pivot

▨ > pivot

Partition



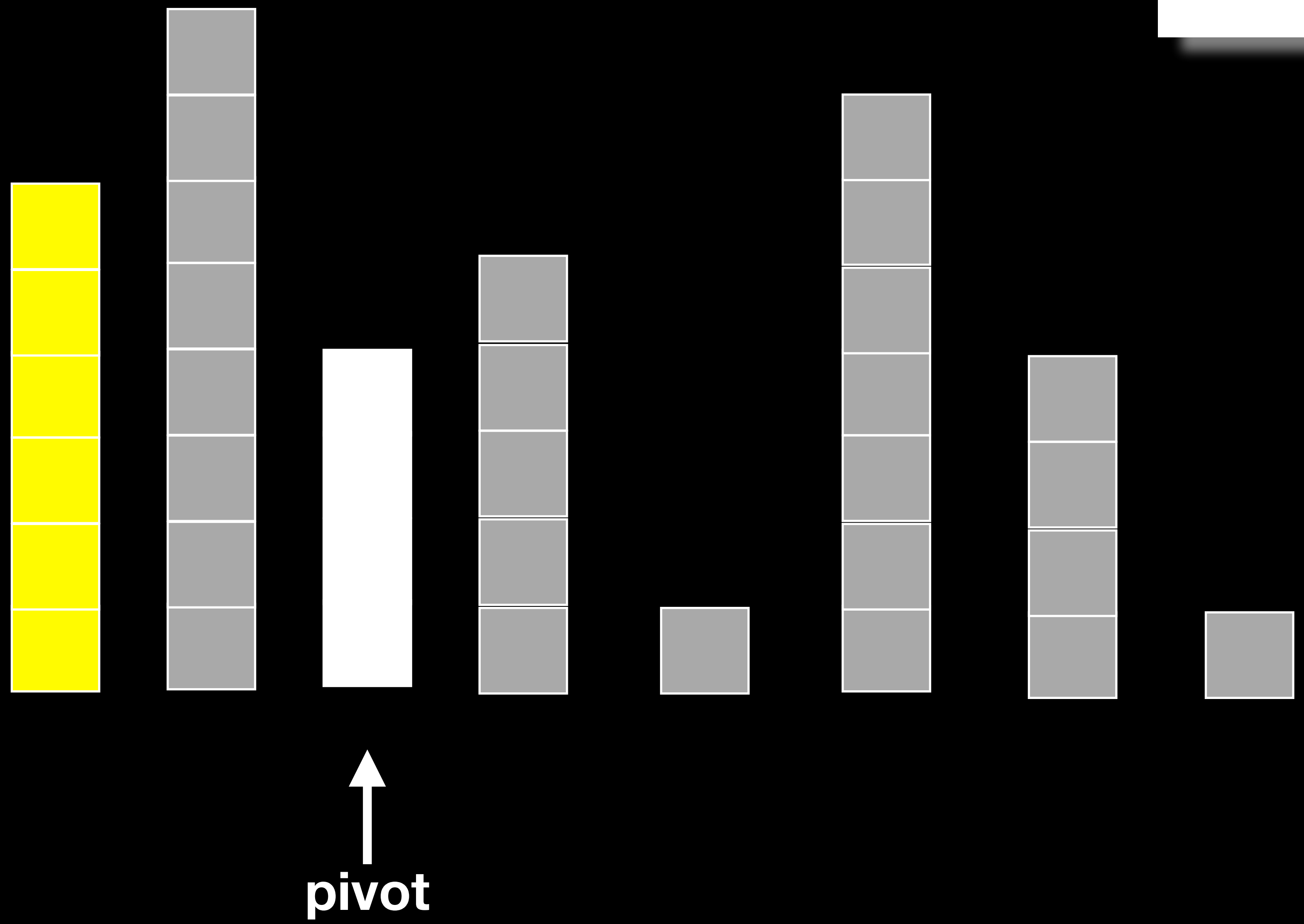
Quick Sort

□ <= pivot

■ > pivot

Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

Partition



Quick Sort

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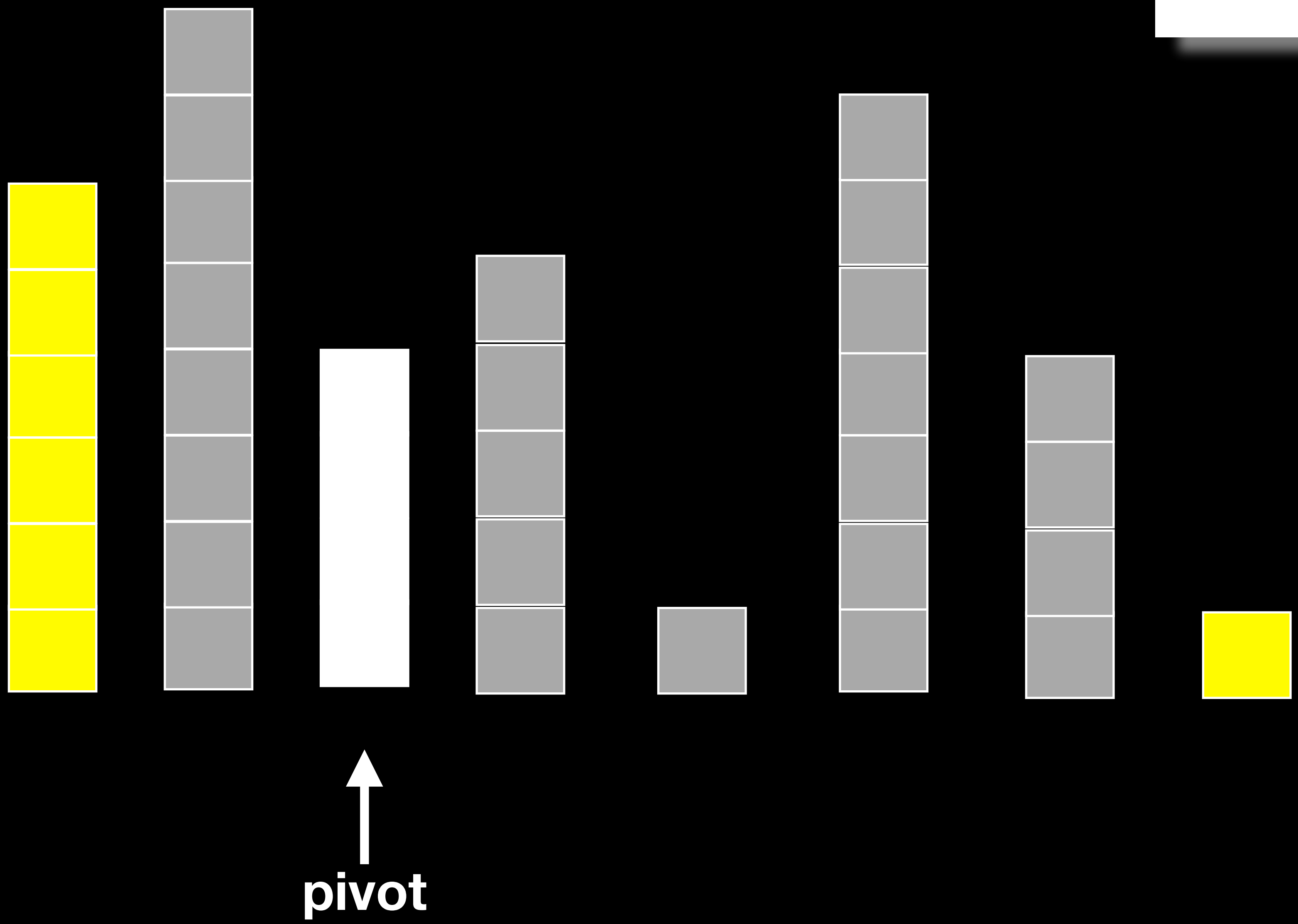
⬜ **<= pivot**

◼ **> pivot**

Partition



Quick Sort

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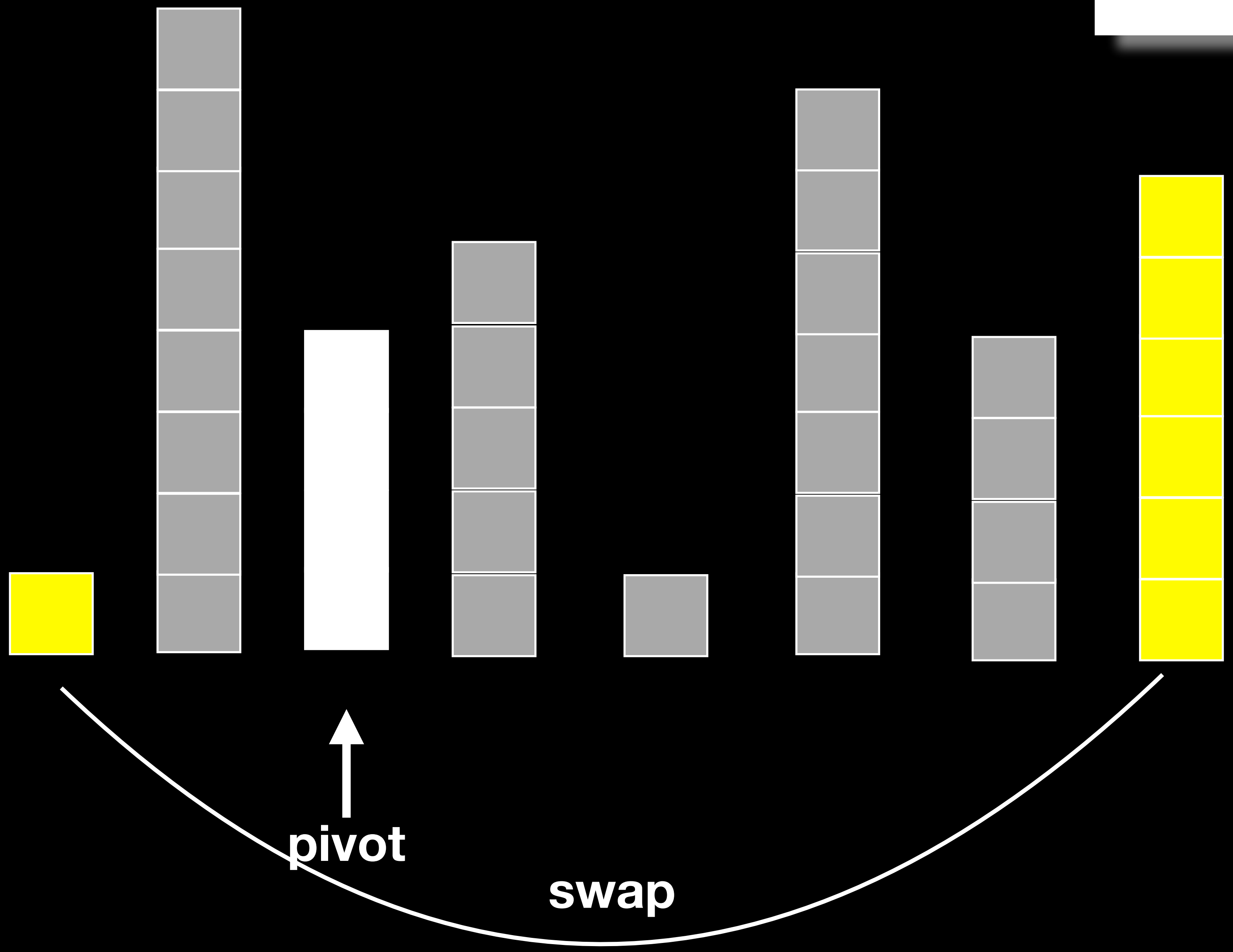
Partition



Quick Sort

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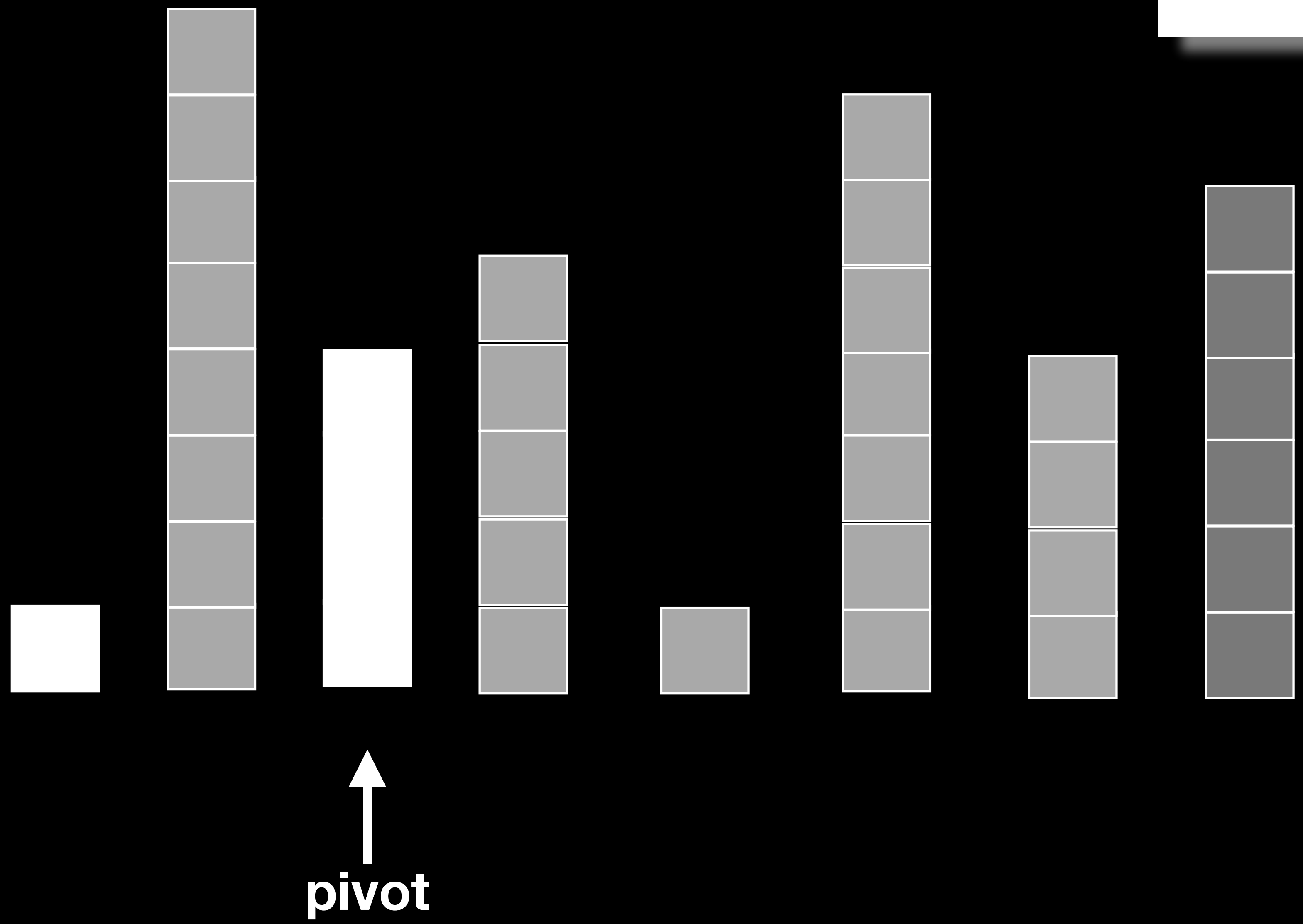
Partition



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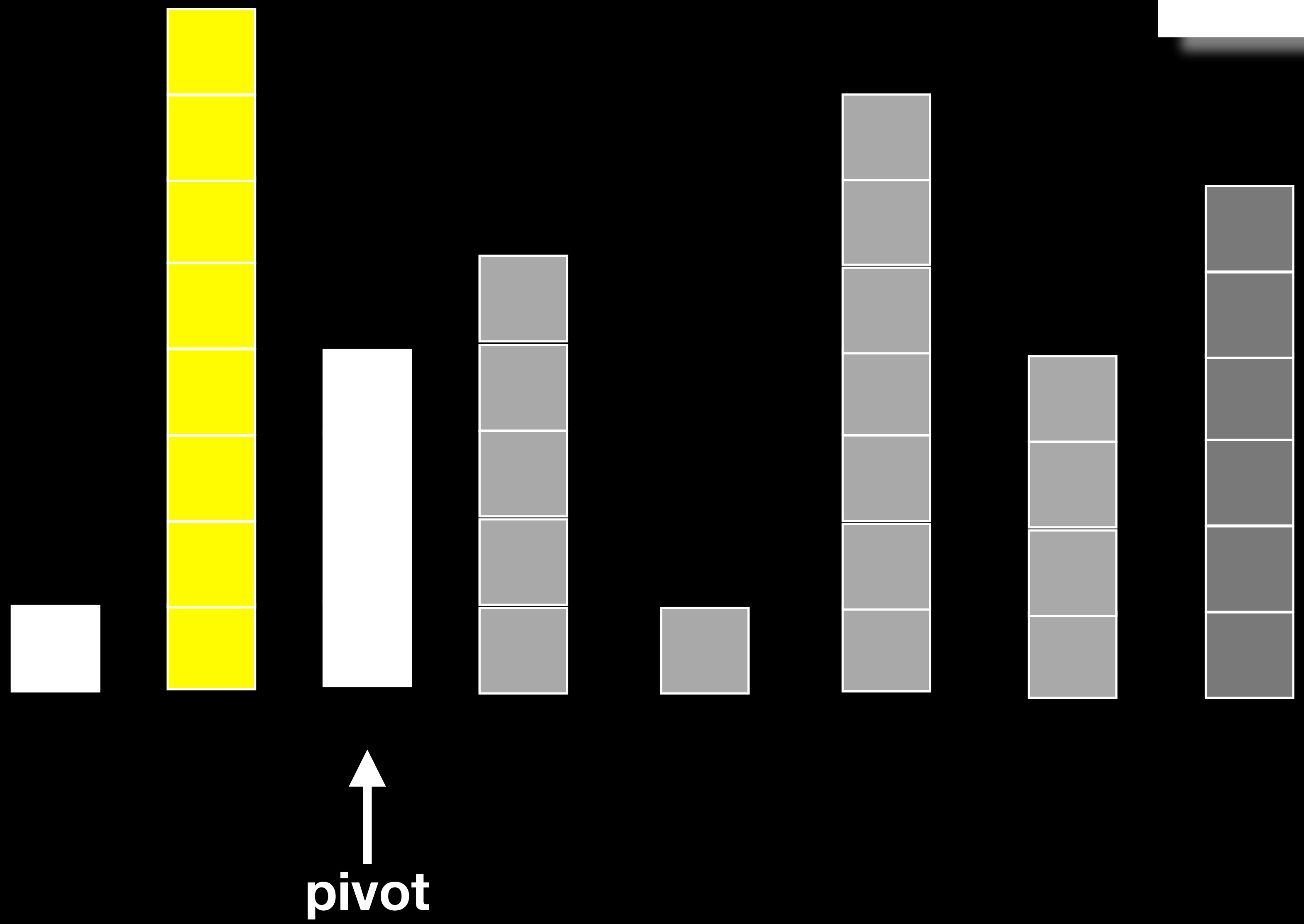
Partition



Quick Sort

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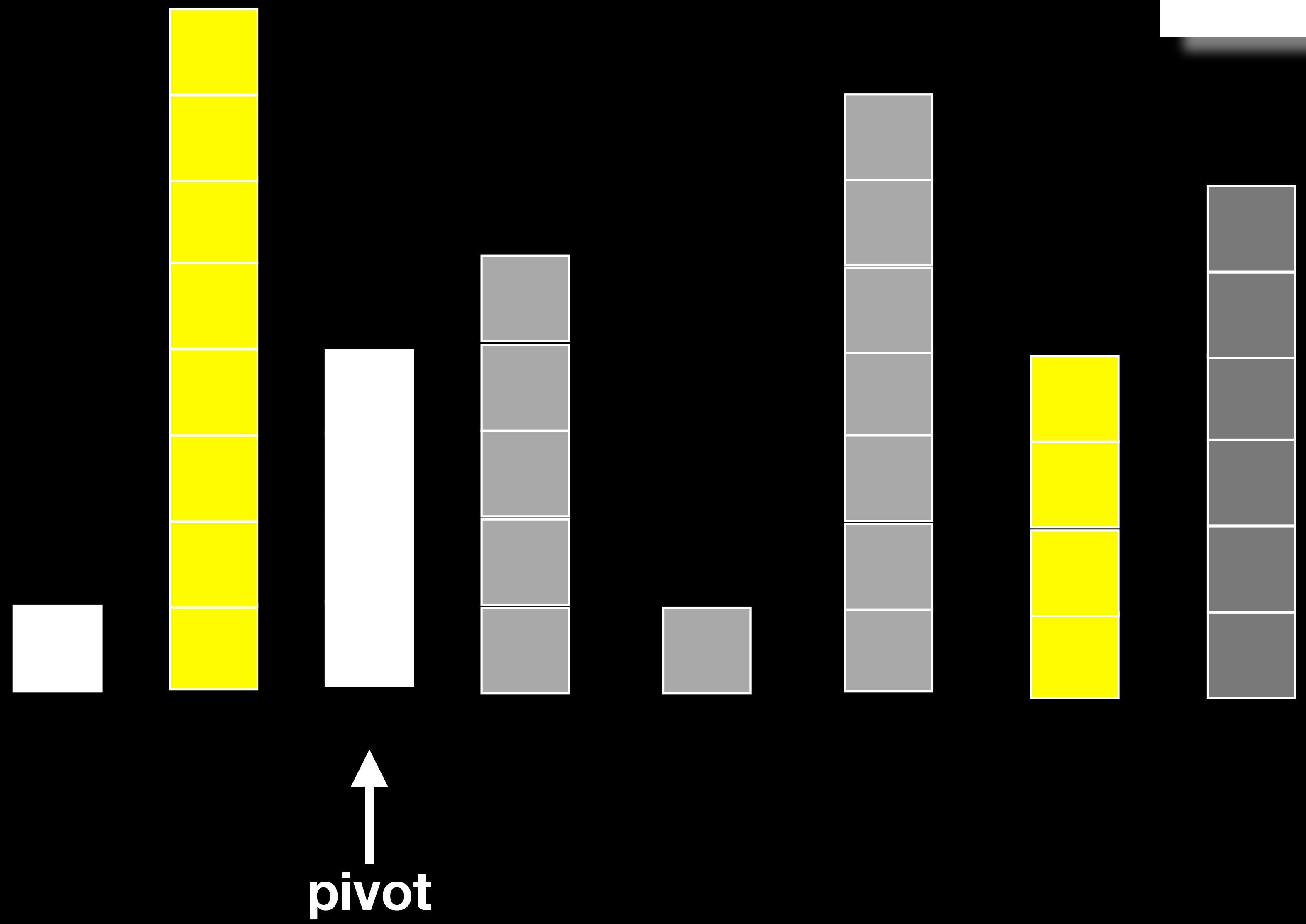
Partition



Quick Sort

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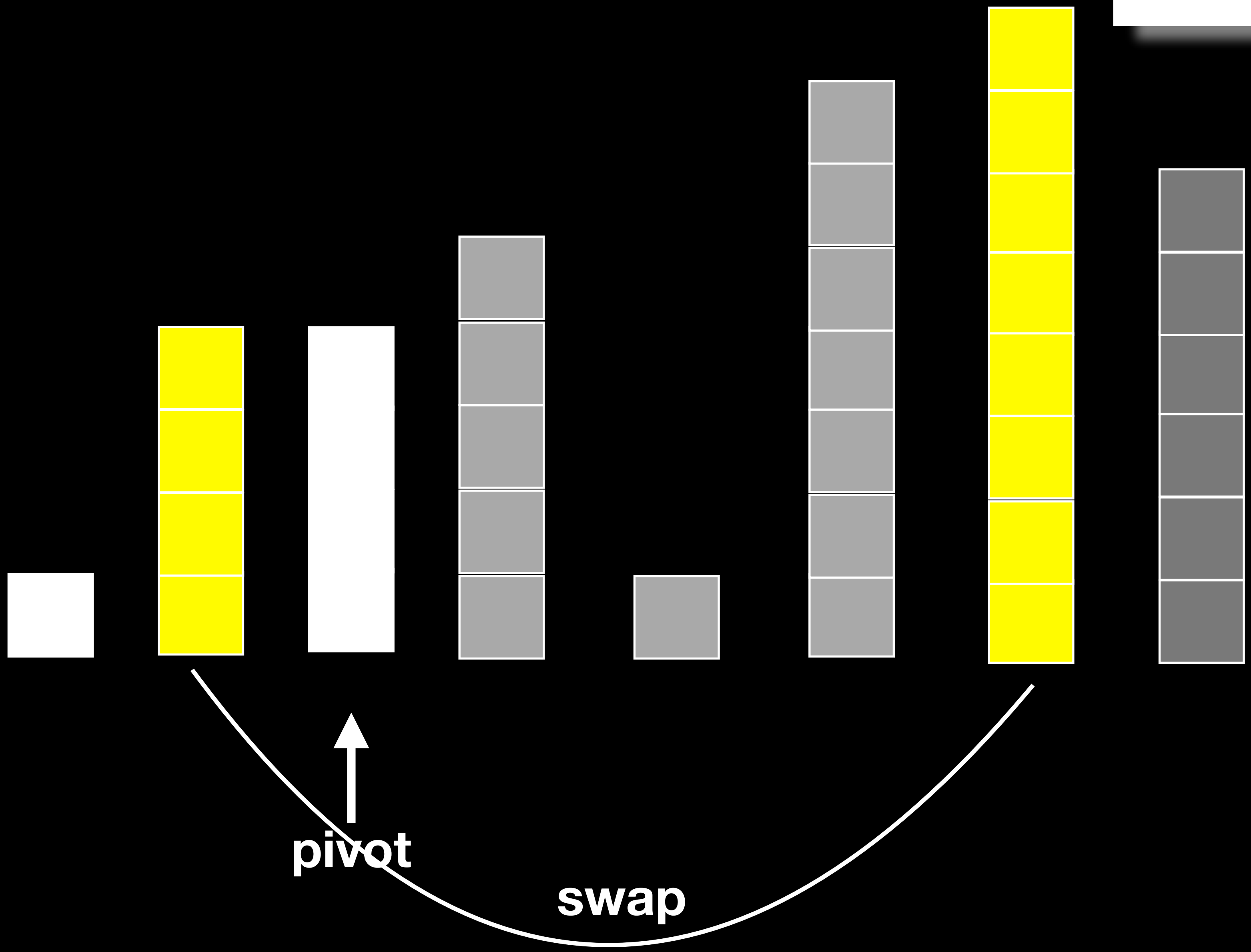
Partition



Quick Sort

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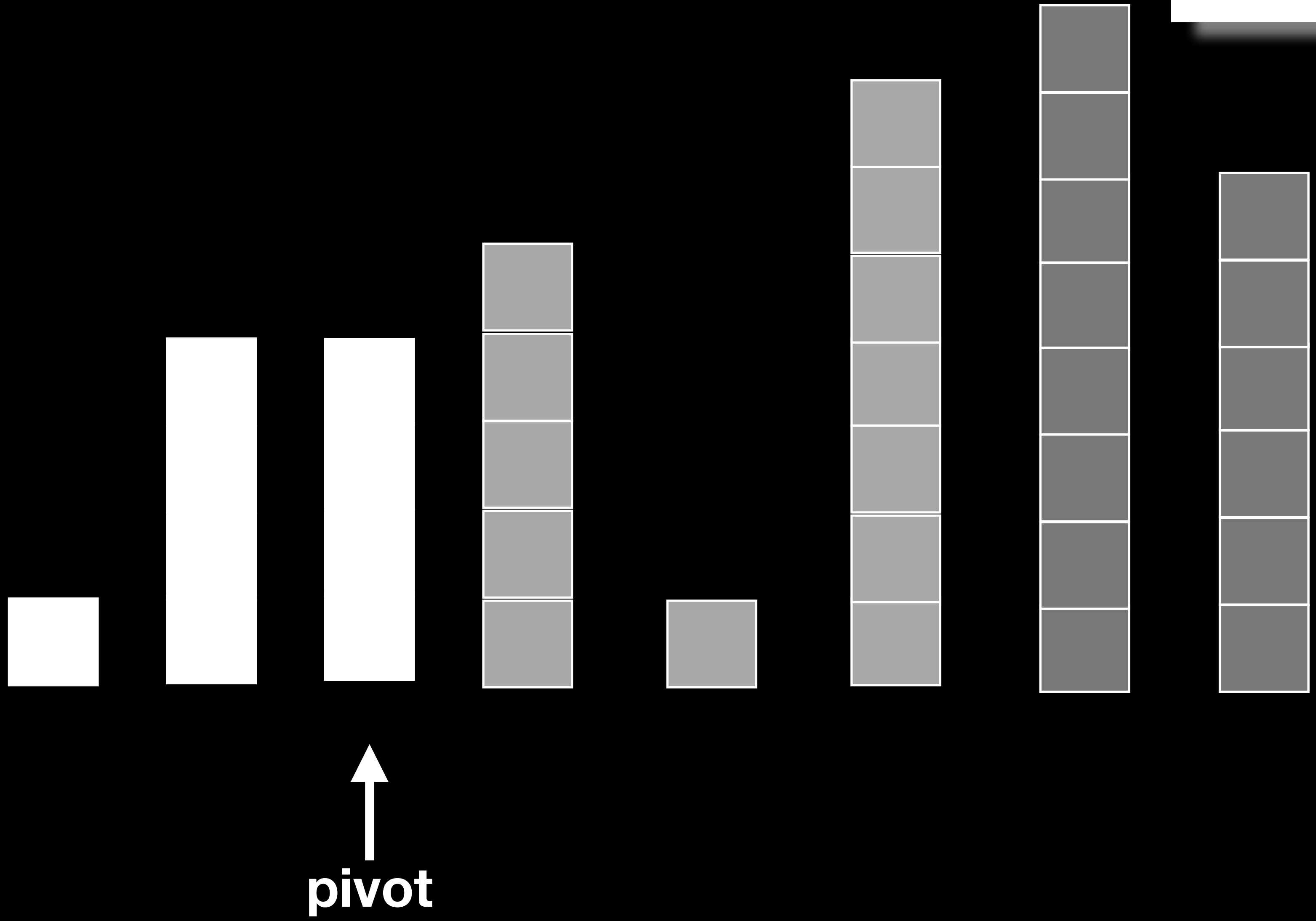
Partition



Quick Sort

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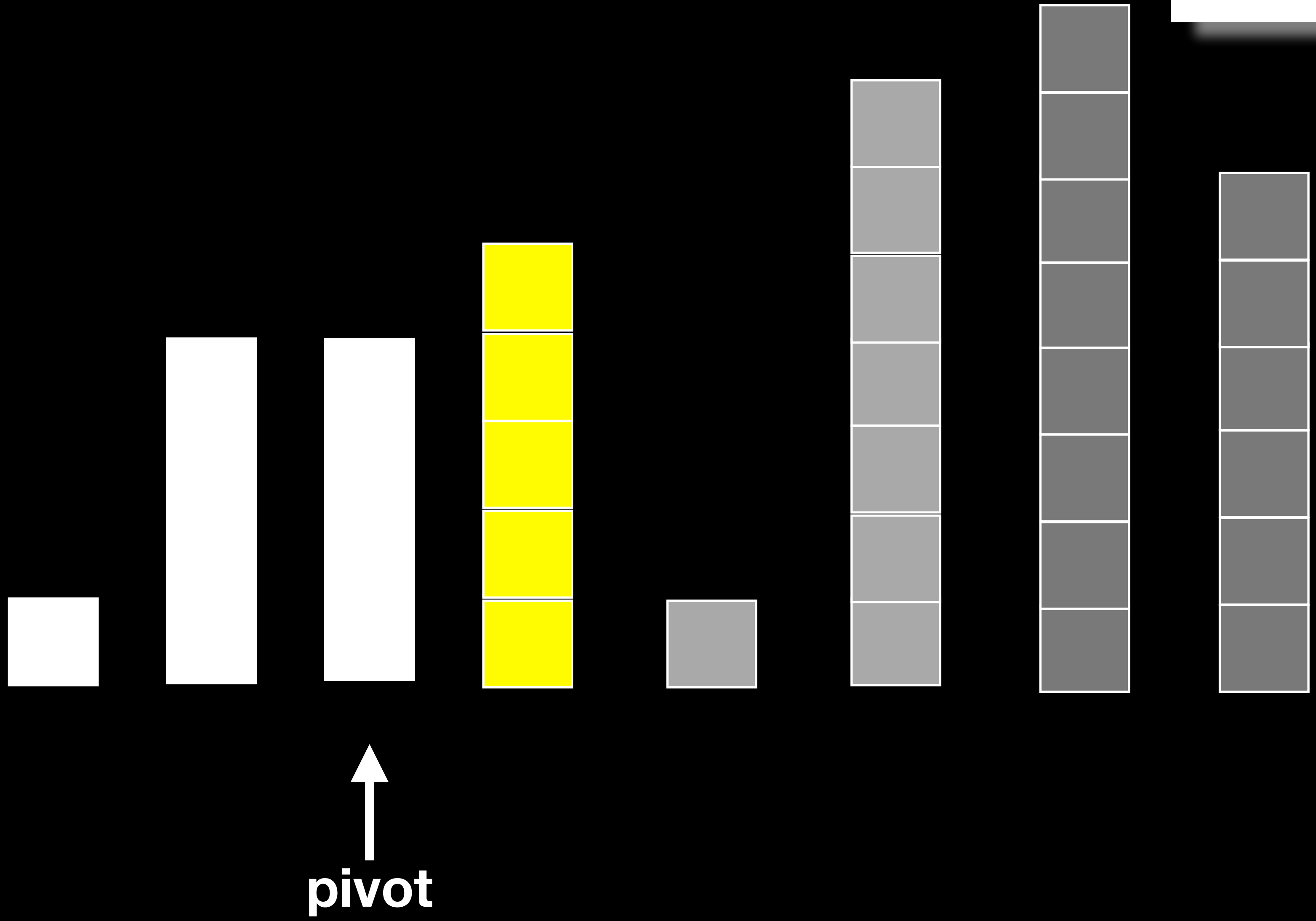
Partition



Quick Sort

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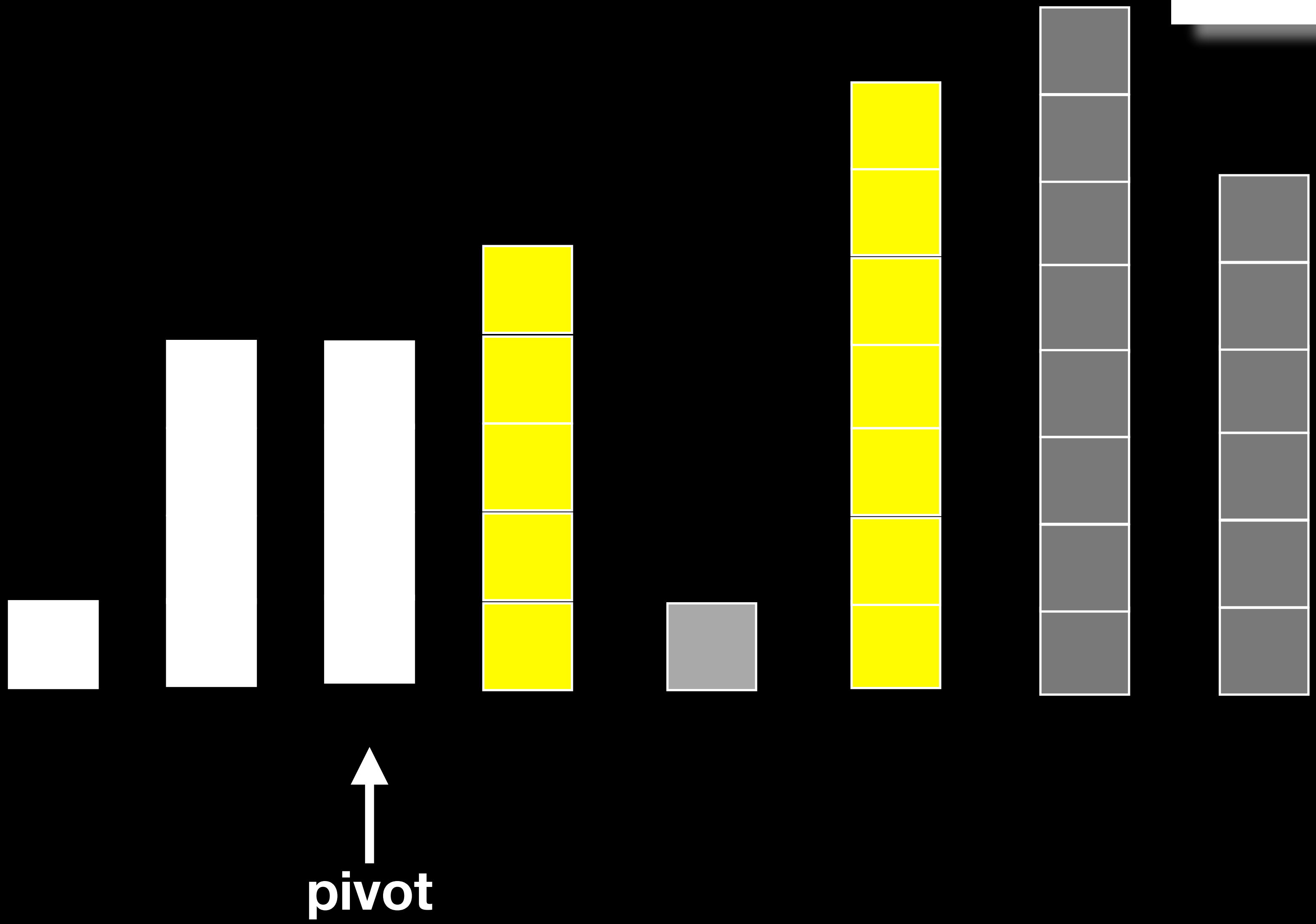
Partition



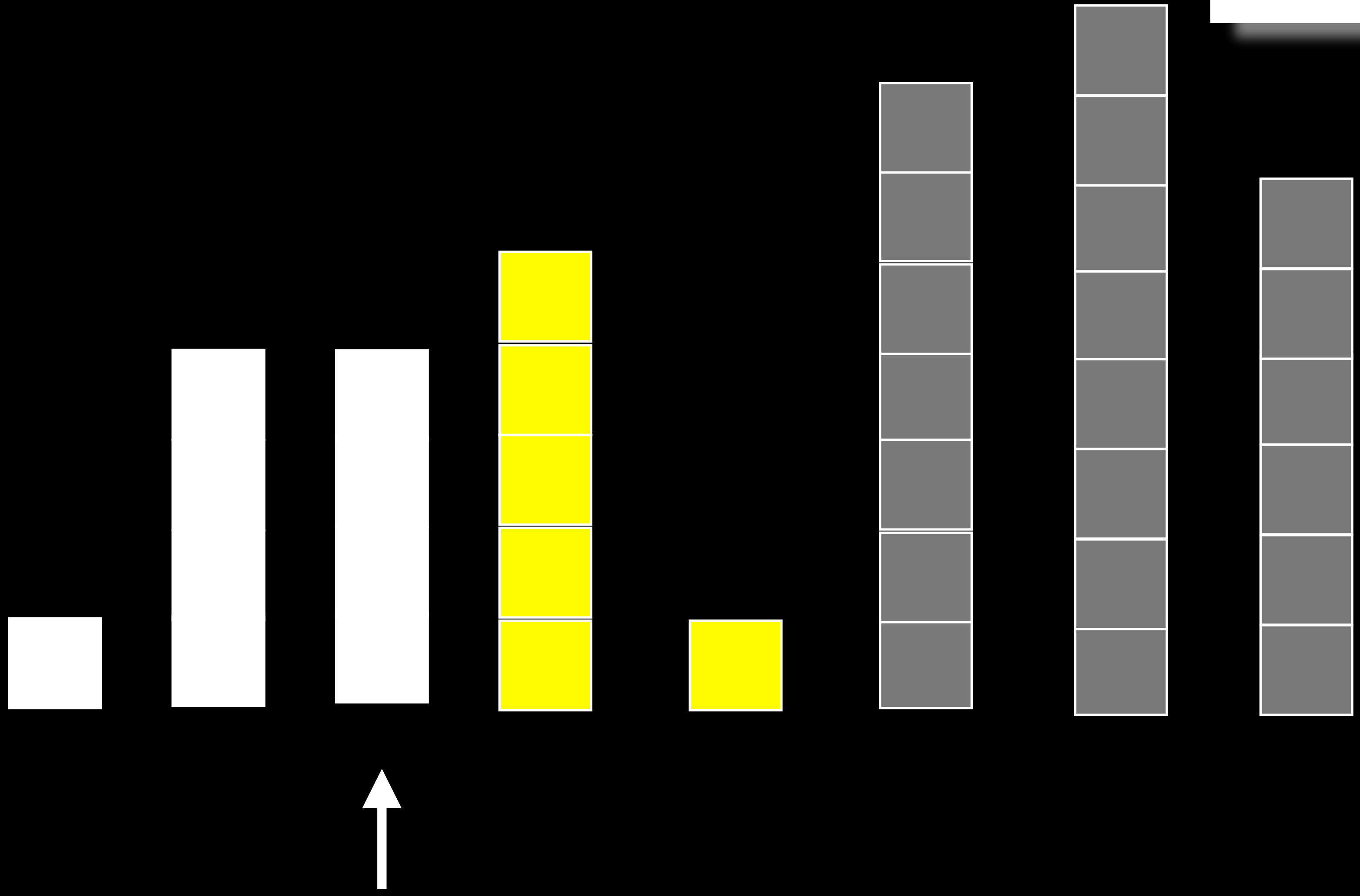
Quick Sort

⬜ <= pivot

⬛ > pivot

Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

Partition



pivot

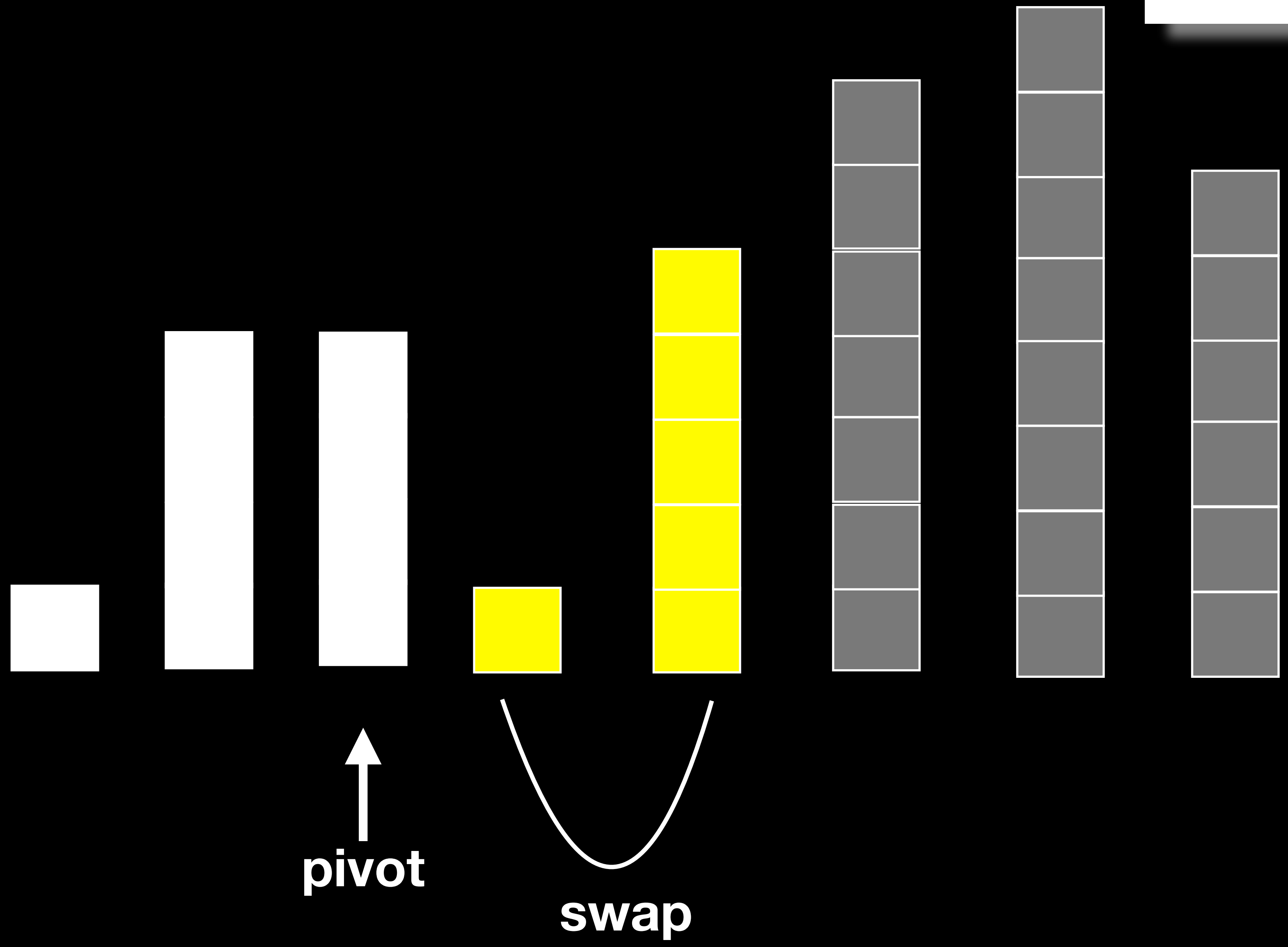
Quick Sort

Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

⬜ <= pivot

▨ > pivot

Partition



Quick Sort

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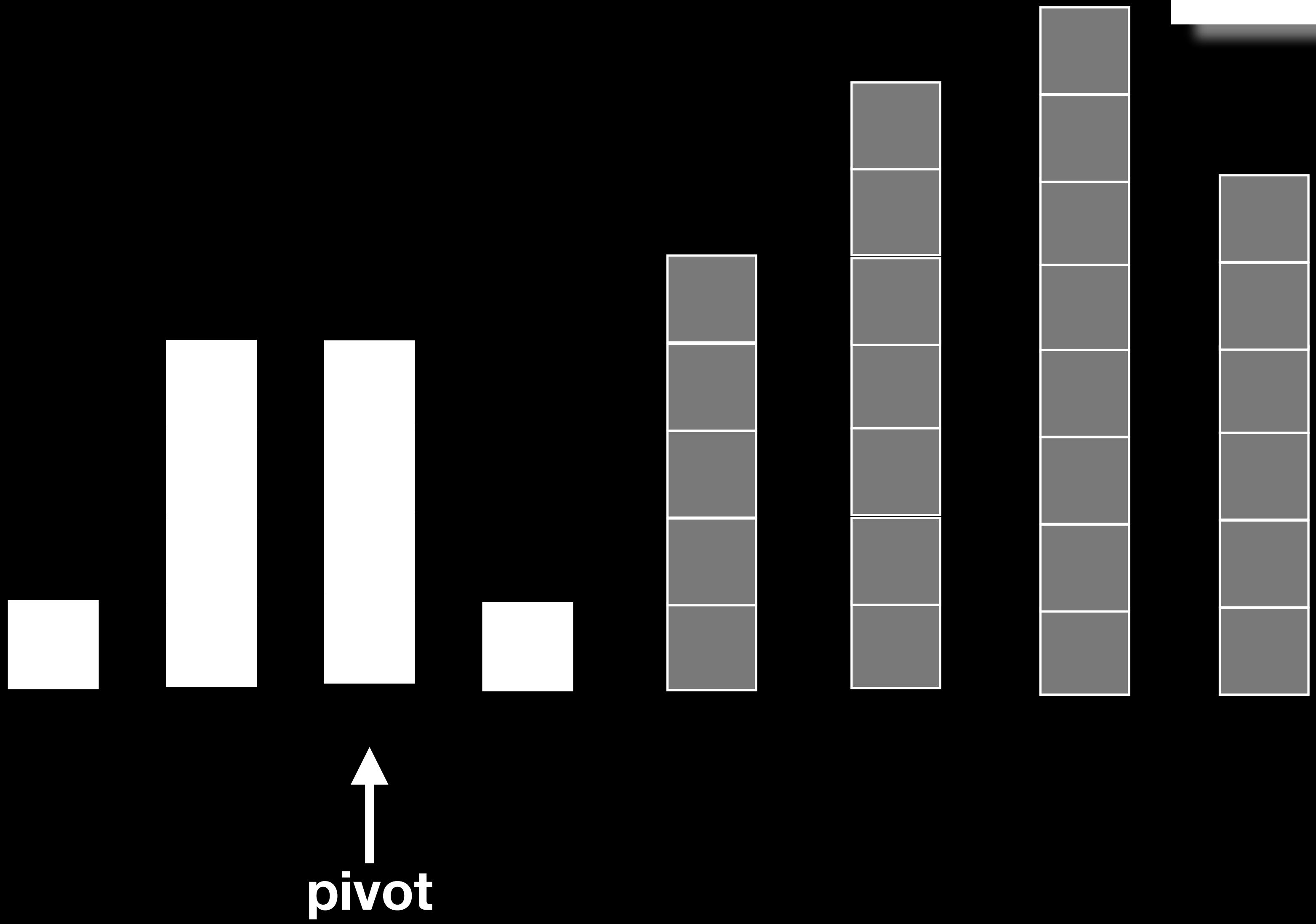
Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

Partition



Quick Sort

⬜ <= pivot

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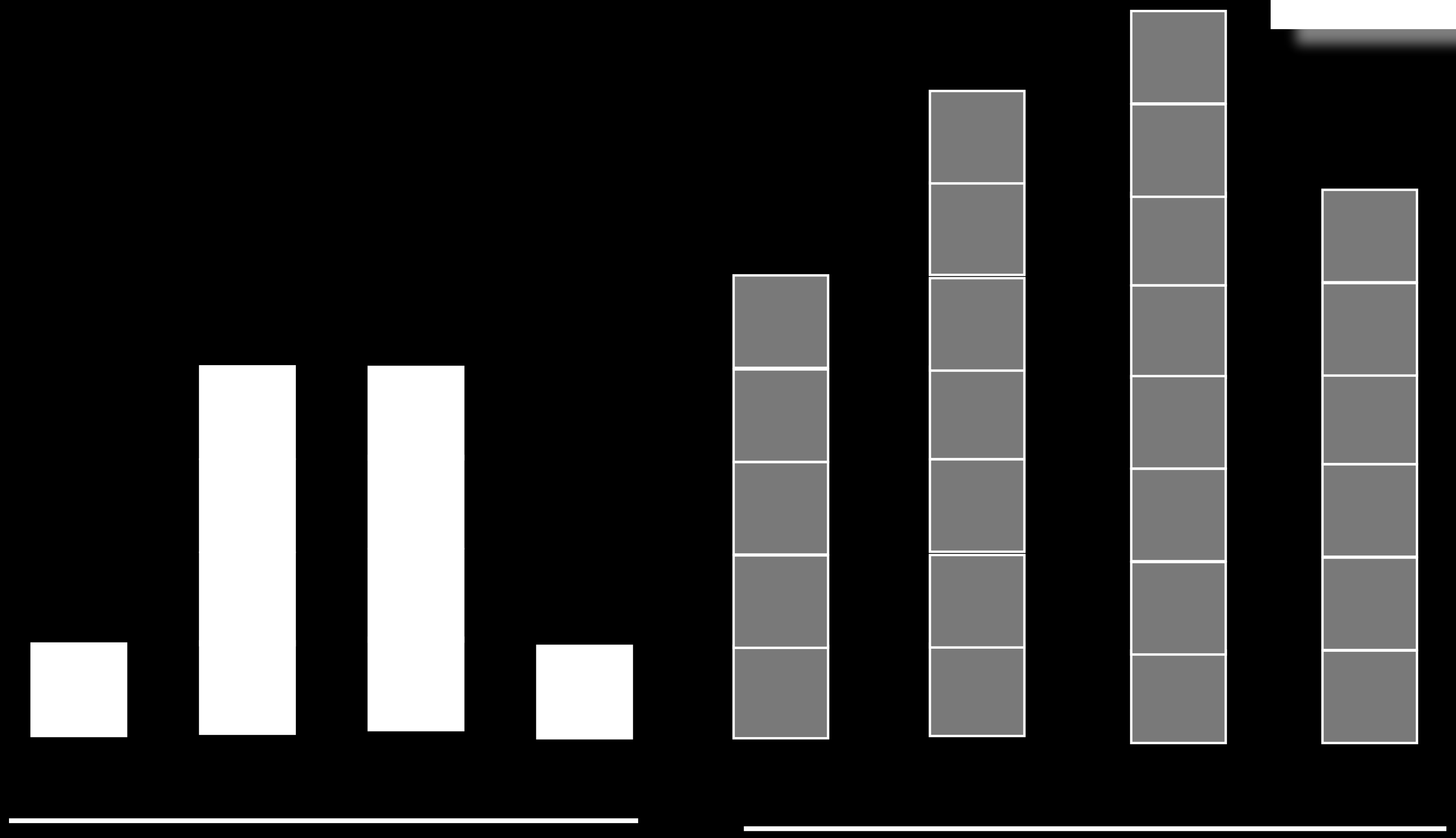
Select a **pivot**. Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot

Partition



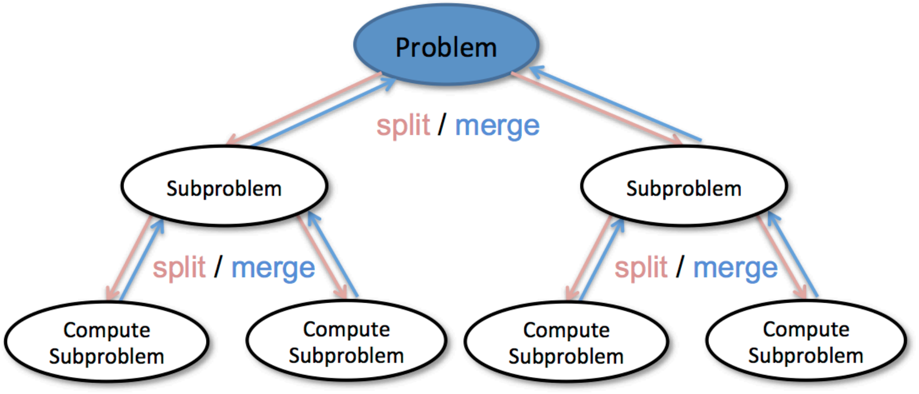
≤ pivot

`quickSort( )`

> pivot

`quickSort( )`

Quick Sort Analysis



Divide and Conquer

n comparisons for each partition

How many subproblems? => Depends on pivot selection

Ideally partition divides problem into two n/2 subproblems for log(n) recursive calls (Best case)

Possibly (though unlikely) each partition has 1 empty subarray for n recursive calls (Worst case)

```
template<typename ItemType>
void quickSort(ItemType the_array[], int first, int last)
{
   if (last - first + 1 < MIN_SIZE)
   {
      insertionSort(the_array, first, last);
   }
   else
   {
      // Create the partition: S1 | Pivot | S2
      int pivot_index = partition(the_array, first, last);

      // Sort subarrays S1 and S2
      quickSort(the_array, first, pivot_index);
      quickSort(the_array, pivotIndex + 1, last);
   }  // end if
}  // end quickSort
```

Optimization

Optimization

How to select pivot?

How to select pivot?

Ideally median

    Need to sort array to find median

Other ideas?

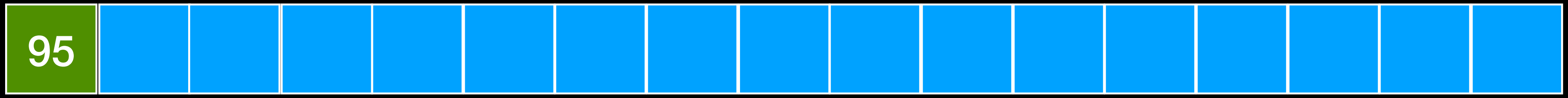
How to select pivot?

Ideally median

  Need to sort array to find median

Other ideas?

  Pick first



How to select pivot?

Ideally median

  Need to sort array to find median

Other ideas?

  Pick first, middle, last position and order them making middle the pivot

95								6								13

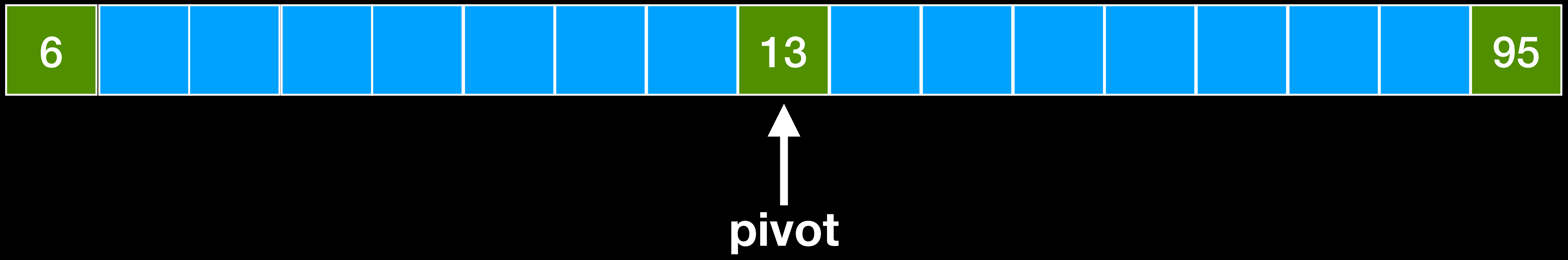
How to select pivot?

Ideally median

Need to sort array to find median

Other ideas?

Pick first, middle, last position and order them making middle the pivot



Quick Sort Analysis

Execution time DOES depend on initial arrangement of data AND on PIVOT SELECTION (luck?) => on random data can be faster than Merge Sort

Possible optimization (e.g. smart pivot selection, speed up base case, iterative instead of recursive implementation) can improve actual runtime -> fastest comparison-based sorting algorithm **on average**

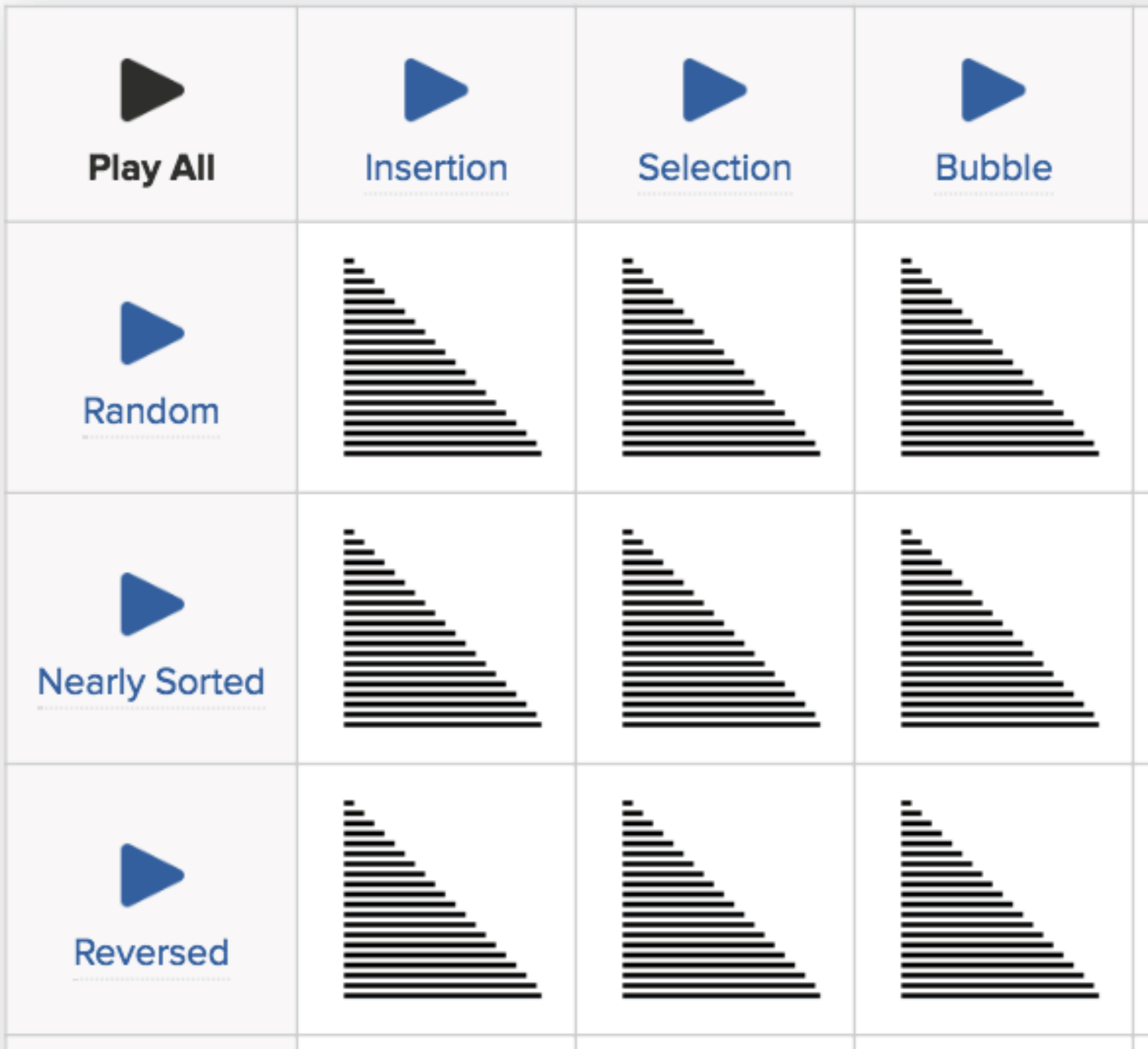
Worst Case: O(n^2) comparisons and data moves

Best Case: O(n log n) comparisons and data moves

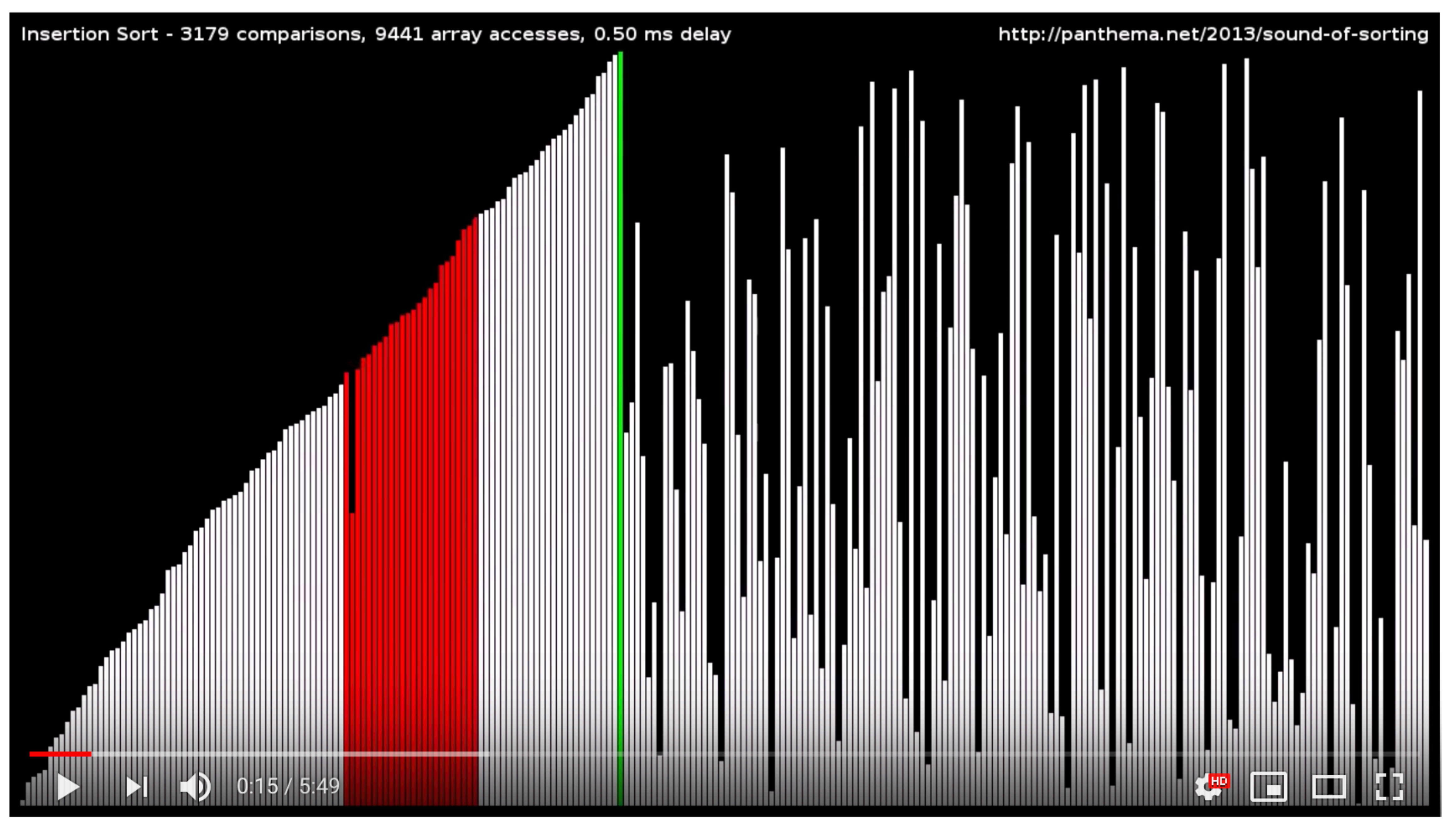
Unstable

	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n^2)$	$O(n)$
Bubble Sort	$O(n^2)$	$O(n)$
Merge Sort	$O(n \log n)$	$O(n \log n)$
Quick Sort	$O(n^2)$	$O(n \log n)$

https://www.toptal.com/developers/sorting-algorithms



https://www.youtube.com/watch?v=kPRA0W1kECg



Insertion Sort - 3179 comparisons, 9441 array accesses, 0.50 ms delay

http://panthema.net/2013/sound-of-sorting